

## TILINGS DEFINED BY ROOT SYSTEMS OF KAC-MOODY ALGEBRAS

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**ABSTRACT.** For root systems of symmetrizable Kac-Moody algebras, we study a tiling of the positive root cone of the form  $\bigcup_{w \in W} (1-w)C^+$ , where  $W$  is the Weyl group and  $C^+$  is the fundamental chamber. We show for general symmetrizable Kac-Moody algebras the tiles are disjoint, and the gaps between top dimensional tiles have codimension  $\geq 2$ . For affine Kac-Moody algebras we completely describe the closure  $\overline{\bigcup_{w \in W} (1-w)C^+}$ .

**RÉSUMÉ.** Pour les systèmes des racines d'algèbres de Kac-Moody symétriques, nous étudions un carrelage du cône des racines positifs de la forme  $\bigcup_{w \in W} (1-w)C^+$ , où  $W$  est le groupe de Weyl et  $C^+$  est la chambre fondamentale. Nous montrons que les carreaux sont disjoints pour les algèbres de Kac-Moody symétriques, et les lacunes entre les carreaux de dimension supérieure ont codimension  $\geq 2$ . Pour les algèbres de Kac-Moody affines, nous décrivons complètement  $\overline{\bigcup_{w \in W} (1-w)C^+}$ .

**1. Introduction** In [5], Waldspurger discovered an interesting decomposition of the positive root cone for root systems of semisimple Lie algebras. Let  $(V, \Pi)$  be a reduced root system with open Weyl chamber  $C^+$ , and Weyl group  $W$  that acts via reflections on  $V$ . Let  $\text{cone}(\Pi)$  denote the closed convex cone generated by the positive roots. Waldspurger proved that  $\text{cone}(\Pi) = \coprod_{w \in W} (1-w)(C^+)$ , which defines a tiling for  $\text{cone}(\Pi)$ . In [4], Meinrenken extended this result to affine Weyl groups, where it is shown that  $V = \coprod_{w \in W} (1-w)D$ . Here  $V$  is the entire Euclidean space,  $W$  is the affine Weyl group, and  $D$  is the open fundamental alcove. In [1], Bibikov and Zghoon simplified the proofs of [4, 5] and extended the result to cocompact discrete hyperbolic reflection groups.

In this note we examine a similar problem for root systems of symmetrizable Kac-Moody algebras. We show such tilings also have image in the closed convex cone generated by the positive roots. We also show the gaps between top dimensional tiles have codimension  $\geq 2$ . For affine Kac-Moody algebras, we completely characterize the union of the closed top dimensional tiles

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**2. Some General Results on Kac-Moody Algebras** We first look at the results that are true for general symmetrizable Kac-Moody algebras. We follow the conventions from Kac's book, [3]. Let  $A = \{a_{ij}\}$  be a symmetrizable generalized Cartan matrix of size  $n$  and rank  $l$ , let the triple  $(\mathfrak{h}, \Pi, \check{\Pi})$  denote a realization:

$$(1) \quad \mathfrak{h} \text{ is a real vector space of dimension } 2n - l,$$

$$(2) \quad \Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^* \text{ is linearly independent,}$$

$$(3) \quad \check{\Pi} = \{\check{\alpha}_1, \dots, \check{\alpha}_n\} \subset \mathfrak{h} \text{ is linearly independent,}$$

$$(4) \quad \langle \alpha_i, \check{\alpha}_j \rangle = a_{ij}, \quad i, j = 1, \dots, n,$$

where  $\Pi$  and  $\check{\Pi}$  denote the root basis and the coroot basis respectively, and  $\langle \cdot, \cdot \rangle$  denotes the natural pairing between  $\mathfrak{h}$  and its dual,  $\mathfrak{h}^*$ .

Let

$$(5) \quad C^+ := \{h \in \mathfrak{h}^*, \langle h, \check{\alpha}_j \rangle > 0\},$$

be the open fundamental chamber. We define, for  $I \subset \{1, \dots, n\}$ , the faces of the fundamental chamber  $C_I^+$  as:

$$(6) \quad C_I^+ := \{h \in \mathfrak{h}^*, \langle h, \check{\alpha}_i \rangle > 0, \langle h, \check{\alpha}_j \rangle = 0, \quad i \in I, j \notin I\}.$$

Let  $\text{cone}(\Pi)$  denote the closed convex cone generated by  $\Pi$ , which is also known as the positive root cone.  $W$  denotes the Weyl group generated by simple root reflections of the form  $s_{\alpha_i}$ , where for  $h \in \mathfrak{h}^*$ :

$$(7) \quad s_{\alpha_i}(h) = h - \langle h, \check{\alpha}_i \rangle \alpha_i.$$

Note that for symmetrizable Kac-Moody algebras the root system is equipped with a non-degenerate invariant bilinear form,  $(\cdot, \cdot)$  (see [3, Chapter 2]). If we use this form to identify  $\mathfrak{h}$  and  $\mathfrak{h}^*$  we have the identity:  $\check{\alpha}_i = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$ .

Let

$$(8) \quad \mathfrak{S} = \{h \in \mathfrak{h}^*, \langle h, \check{\alpha}_j \rangle = 0, j = 1, \dots, n\}.$$

Recall from (1) that  $\mathfrak{h}^*$  has dimension  $2n - l$ . By (2), we then have  $\dim \mathfrak{S} = n - l$ .

We consider the maps, for  $w \in W$ :

$$(9) \quad 1 - w : \mathfrak{h}^* \rightarrow \mathfrak{h}^*.$$

Observe that the Weyl group restricts to the identity on  $\mathfrak{S}$ , so there is an induced Weyl group action on:

$$(10) \quad \widetilde{\mathfrak{h}}^* := \mathfrak{h}^*/\mathfrak{S}.$$

Therefore  $1 - w$  induces a map:

$$(11) \quad \phi_w : \widetilde{\mathfrak{h}}^*/\mathfrak{S} \cong \text{span} \{ \check{\Pi} \}^* \rightarrow \text{span} \{ \Pi \}.$$

Here we can take image to be  $\text{span} \{ \Pi \}$  instead of  $\mathfrak{h}^*$  because of Equation (7) (or more directly by Equation (12) below), then  $\phi_w$  becomes a map between vector spaces of the same dimension.

Consider the image of  $C^+$  under the quotient map:  $\widetilde{C}^+ := C^+/\mathfrak{S}$ . Observe  $\widetilde{C}^+$  is a closed pointed convex polyhedral cone. Then we define the tiles  $C_w := \phi_w(\widetilde{C}^+) \subset \text{span} \{ \Pi \} \subset \mathfrak{h}^*$ . We say a tile is top dimensional if  $\phi_w$  is invertible. Note since for  $w \in W$ ,  $1 - w$  restricts to 0 on  $\mathfrak{S}$ , we have the equality  $C_w = (1 - w)(C^+)$  as well.

**THEOREM 2.1.** *For  $w \in W$ , the tiles  $C_w$  are mutually disjoint, and they lie in  $\text{cone}(\Pi)$ .*

**PROOF.** The technique is from [5]. We have that if  $w \in W$  is given by  $w = s_{\alpha_{i_1}} \dots s_{\alpha_{i_n}}$  as reduced expression, a simple induction shows that:

$$(12) \quad (1 - w)(h) = \sum_{k=1, \dots, n} \langle h, \check{\alpha}_{i_k} \rangle s_{\alpha_{i_1}} \dots s_{\alpha_{i_{k-1}}}(\alpha_{i_k}).$$

From [3, Chapter 3],  $s_{\alpha_{i_1}} \dots s_{\alpha_{i_{k-1}}}(\alpha_{i_k}) \in \text{cone}(\Pi)$  if  $w$  is reduced. Furthermore, if  $h \in C^+$ ,  $\langle h, \check{\alpha}_{i_k} \rangle > 0$ . In particular, this proves  $C_w = (1 - w)C^+ \subset \text{cone}(\Pi)$ .

To prove disjointedness, assume to the contrary that not all tiles are disjoint. Then there are  $v, v' \in C^+$ ,  $w, w' \in W$  such that  $(1 - w)v = (1 - w')v'$ . Let  $y = w^{-1}w'$ , then we have  $v - v' = w(v - y(v'))$ . Using the non-degenerate invariant bilinear form  $(\ , \ )$ , we get:

$$(13) \quad \begin{aligned} (v - v', v - v') &= \left( w(v - y(v')), w(v - y(v')) \right) \\ &= (v - y(v'), v - y(v')) \\ &= (v, v) + (v', v') - 2(v, y(v')). \end{aligned}$$

Expanding the left hand side of Equation (13) and collecting like terms:

$$(14) \quad (v, v' - y(v')) = 0.$$

However,  $v \in C^+$ , and from the first part of the theorem  $v' - y(v') \in \text{cone}(\Pi)$ . This contradicts (14) unless  $y(v') = v'$ , in which case  $y = 1$  (the Weyl group acts without stabilizer in  $C^+$ , see [3, Chapter 3]). Hence we have  $w = w'$ .  $\square$

Recall (see [3, Chapter 5]) that a root is real if and only if it is the image of a simple positive root (element of  $\Pi$ ) under the action of the Weyl group. A root is imaginary if it is not real. From [3, Chapter 5] we see that a root  $\alpha$  is imaginary if and only if  $(\alpha, \alpha) \leq 0$ .

**PROPOSITION 2.2.** *Let  $\beta_1, \dots, \beta_p$  be linearly independent real roots, then  $w = s_{\beta_1} \dots s_{\beta_p} \in W$ , and  $\dim C_w = p$ .*

**PROOF.** The fact that  $w \in W$  follows from [3, Chapter 5]. The rest follows from Equation (12) and the observation that the set  $\{\beta_1, s_{\beta_1}\beta_2, \dots, s_{\beta_1} \dots s_{\beta_{p-1}}\beta_p\}$  is linearly independent.  $\square$

In [5], it is proved that for root systems of finite semisimple Lie algebras the tiling is exact, i.e.  $\text{cone}(\Pi) = \coprod_{w \in W} (1-w)(C^+)$ . However, we observe the result is not true for more general symmetrizable Kac-Moody algebras. In particular, we observe that the tiling can never cover the imaginary roots.

**PROPOSITION 2.3.**  *$C_w$  cannot contain any imaginary roots.*

**PROOF.** For all  $h \in C^+$ , we have that  $(h - wh, h - wh) = 2(h, h - wh)$ , using the invariance of the bilinear form. If  $h - wh = 0$  the conclusion holds trivially; if  $h - wh \neq 0$ , then  $2(h, h - wh) > 0$  since  $h \in C^+$  and  $(1 - w)h \in \text{cone}(\Pi)$ . Hence  $h - wh$  cannot be an imaginary root.  $\square$

Recall (see [3, Chapter 4]) that Kac-Moody algebras are classified into 3 types: finite, affine, and indefinite. We also have that Kac-Moody algebras of indefinite type have positive imaginary roots of negative length. In particular, the above result implies that for indefinite types not only is the tiling not surjective, but even if we take the closure of the tiling, it still cannot cover the positive root cone, since vectors near the imaginary roots of negative length will also have negative length.

**Example 1** The root system for the affine Kac-Moody algebra  $A_1^{(1)}$  can be obtained by adjoining a simple root to the root system of the simple Lie algebra  $A_1$ . Its tiling is given below, with the first 200 tiles shown. It is possible to show directly that all of the positive root cone is covered except the open half line spanned by the imaginary root. The tiling is otherwise exact, i.e. no lower dimensional gaps of any kind between the tiles. In other words we have  $\text{cone}(\Pi) \setminus \mathbb{R}_+\delta = \coprod_{w \in W} C_w$ .

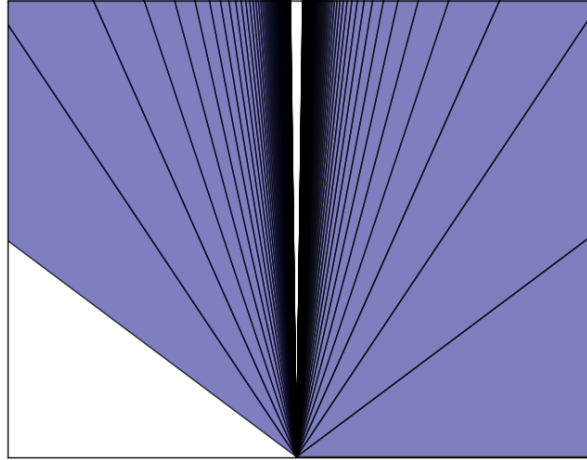


Figure 1: Tiling of the root system  $A_1^{(1)}$ .

Let  $W_{reg}$  denote all  $w \in W$  such that  $C_w$  is top dimensional. Observe if  $w \in W_{reg}$ , then  $w^{-1} \in W_{reg}$ . We first observe that top dimensional tiles exist.

PROPOSITION 2.4. *Let  $w = s_{\alpha_1} \dots s_{\alpha_n}$  denote the Coxeter element, then  $C_w$  is top dimensional.*

PROOF. We follow closely the proof in [2, section 3.16]. It suffices to show that the fixed point set of  $w$  in  $\mathfrak{h}^*$  is  $\mathfrak{S}$ . To this end, consider  $v \in \mathfrak{h}^*$  such that  $wv = v$ . It follows that:

$$(15) \quad s_{\alpha_2} \dots s_{\alpha_n} v = s_{\alpha_1} v.$$

The right hand side of (15) is of the form  $v + c_1 \alpha_1$ , whereas the left hand side is of the form  $v + c_2 \alpha_2 + \dots + c_n \alpha_n$ . Since  $\Pi$  is a linear independent set, we must have both sides are equal to  $v$ . This shows right hand side is  $v$ , hence  $c_1 = 0$  and therefore  $\langle v, \check{\alpha}_1 \rangle = 0$ . We are now left with the equation:

$$(16) \quad s_{\alpha_3} \dots s_{\alpha_n} v = s_{\alpha_2} v$$

A similar argument shows  $\langle v, \check{\alpha}_2 \rangle = 0$ . Continuing in this fashion we have  $\langle v, \check{\alpha}_j \rangle = 0$ ,  $j = 1, \dots, n$ , and hence  $v \in \mathfrak{S}$ .  $\square$

Next we investigate how top dimensional tiles fit together.

PROPOSITION 2.5. *Let  $w \in W_{reg}$ , and  $s_i$  the simple reflection corresponding to  $\alpha_i$ . Then  $\dim C_{ws_i} = n - 1$ . Moreover,  $C_{ws_i}$  contains the  $i$ -th open face of  $C_w$ .*

PROOF. See also [5]. Observe  $\mathfrak{S} \subseteq \ker(\tilde{\alpha}_i) \subseteq \mathfrak{h}^*$ , hence  $\ker(\tilde{\alpha}_i)$  descends to  $H_i \subset \tilde{\mathfrak{h}}^*$ . Now  $1 - ws_i$  agrees with  $1 - w$  on  $H_i$ , thus  $\dim C_{ws_i} \geq n - 1$ . Thus it suffices to show  $\phi_{ws_i}$  has nontrivial kernel in  $\tilde{\mathfrak{h}}^*$ . Consider  $\bar{a} \in \mathfrak{h}^* \setminus \mathfrak{S}$  such that  $(1 - w^{-1})(\bar{a}) = \alpha_i$  (possible since  $w \in W_{reg}$ ). Then by direct computation we have:

$$(17) \quad (\bar{a}, \alpha_i) = \frac{(\alpha_i, \alpha_i)}{2} > 0.$$

From this it follows that:

$$(18) \quad \begin{aligned} (1 - ws_i)(\bar{a}) &= \bar{a} - w(\bar{a} - \alpha_i) \\ &= (1 - w)\bar{a} + w(\alpha_i) \\ &= w(w^{-1} - 1)\bar{a} + w(\alpha_i) = 0. \end{aligned}$$

Finally, if we let  $\widetilde{C}_i^+$  be the  $i$ -th open face of  $\widetilde{C}^+$ , observe that  $\phi_w(\widetilde{C}_i^+) = \phi_{ws_i}(\widetilde{C}_i^+)$ . Therefore  $C_{ws_i}$  contains the  $i$ -th open face of  $C_w$ .  $\square$

Now suppose  $u \in W$  such that  $\phi_u$  has corank 1, then let  $N \in \tilde{\mathfrak{h}}^*$  span the kernel of  $\phi_u$ . Let  $\xi \in C_u$  be such that it is not in the image of any codimension two (or higher) face of  $\widetilde{C}^+$ , i.e.  $\xi \notin \phi_u(\widetilde{C}_I^+)$ ,  $|I| \geq 2$  (possible since  $\dim C_u = n - 1$ ). Then  $\phi_u^{-1}(\xi)$  is an affine line  $l$  in the direction of  $N$ .

PROPOSITION 2.6. *The line  $l$  meets the boundary  $\widetilde{C}^+$  in at least one and at most two distinct open codimension 1 faces. If the line  $l$  intersects  $\widetilde{C}_i^+$ , let  $s_i$  be the corresponding root reflection, then  $us_i \in W_{reg}$ .*

PROOF. To see the first claim, by definition of  $\xi$  there exists  $r \in \widetilde{C}^+ \cap \phi_u^{-1}\xi$ , then since  $N \neq 0 \in \tilde{\mathfrak{h}}^*$  there exists some  $\tilde{\alpha}_k \in \tilde{\Pi}$  such that  $\langle N, \tilde{\alpha}_k \rangle \neq 0$ . Then the line  $l(x) = r + xN$  must meet the boundary of  $\widetilde{C}^+$  in at least one and at most two distinct codimensional 1 faces (codimension 1 by definition of  $\xi$ ).

For the second part (also see [5]), let us show  $\phi_{us_i}$  is invertible. First we use the invariant bilinear form to identify  $\mathfrak{h}$  with  $\mathfrak{h}^*$  and lift  $N$  to be an element of  $\mathfrak{h}^*$ , also denoted by  $N$ . Observe since  $l$  meets  $\widetilde{C}_i^+$ ,  $\langle N, \tilde{\alpha}_i \rangle \neq 0$  and hence  $\langle N, \alpha_i \rangle \neq 0$ . Let  $h \in \mathfrak{h}^*$  and let  $h' \in \tilde{\mathfrak{h}}^*$  be its image under the quotient. Assume  $\phi_{us_i}h' = 0$ , which is equivalent to  $us_ih = h$ . Then  $s_ih = u^{-1}h$ , so

$$(19) \quad \phi_{u^{-1}}(h') = (1 - u^{-1})h = (1 - s_i)h = \langle h, \tilde{\alpha}_i \rangle \alpha_i$$

Pairing Equation (19) with  $N$  via the bilinear form, using  $(1 - u)(N) = 0$  and  $\langle \alpha_i, N \rangle \neq 0$  we get  $\langle h, \tilde{\alpha}_i \rangle = 0$ , thus  $\phi_{u^{-1}}h' = 0$ , hence  $\phi_u(h') = 0$ . By definition of  $N$ ,  $h$  is proportional to  $N$  modulo  $\mathfrak{S}$ . However  $\langle h, \alpha_i \rangle = 0$  while  $\langle N, \alpha_i \rangle \neq 0$ , thus we conclude  $h \in \mathfrak{S}$ , and  $h' = 0$ . Therefore  $\phi_{us_i}$  is invertible.  $\square$

To describe more globally how the tiles fit together, we deduce the following proposition, following [5].

PROPOSITION 2.7. *Let  $v \in \text{int}(\text{cone}(\Pi))$ , let  $g \in W_{\text{reg}}$ . Consider the set  $S \subset \text{span}\{\Pi\}$  defined to be:*

$$(20) \quad S := \bigcup_{w \in W} \phi_w(\widetilde{C_I^+}), \quad |I| \geq 2.$$

*Pick  $v_0 \in C_g$  such that the line  $T$  connecting  $v$  and  $v_0$  does not intersect  $S$  (possible since  $S$  has codimension at least 2) and  $v - v_0 \in \text{cone}(\Pi)$ . Further suppose that  $T$  only intersects with finitely many top dimensional tiles. Then  $T \subseteq \bigcup_{w \in W_{\text{reg}}} \overline{C_w}$*

PROOF. We wish to use the invariant bilinear form so it is more useful to consider our domain to be  $C^+$  instead of  $\widetilde{C^+}$ . We proceed by contradiction. For  $t \in [0, 1]$ , let  $v_t = (1 - t)v_0 + tv$ . Let  $\Delta < 1$  denote the largest number such that  $v_t \in \bigcup_{w \in W_{\text{reg}}} \overline{C_w}$  for  $t \in [0, \Delta]$ . Since the intersection of  $T$  with the set of  $C_w$ ,  $w \in W_{\text{reg}}$  is finite, there exists  $\epsilon > 0$  such that:

$$(21) \quad t \in (\Delta, \Delta + \epsilon) \Rightarrow v_t \notin \bigcup_{w \in W_{\text{reg}}} \overline{C_w},$$

$$(22) \quad t \in (\Delta - \epsilon, \Delta) \Rightarrow \exists w \in W_{\text{reg}} \text{ such that } v_t \in C_w.$$

Let  $f := v - v_0$  and  $e, u \in \mathfrak{h}^*$  be such that  $(1 - w)e = f$ ,  $(1 - w)u = v_\Delta$ . Since  $T$  does not intersect  $S$ ,  $u$  is on a codimension 1 face of  $C^+$ : i.e. there exists unique  $\alpha \in \Pi$  such that  $(\alpha, u) = 0$ . By (21) and (22),  $(\alpha, e) < 0$ . Consider  $\sigma := ws_\alpha$ . By Proposition 2.5,  $\phi_\sigma$  has corank 1. Let  $\bar{a} \in C^+$  be such that  $(1 - w^{-1})\bar{a} = \alpha$ . From Proposition 2.5,  $(1 - \sigma)\bar{a} = 0$ . Consider the line  $l(x) = u + x\bar{a}$ , by Equation (17), for small enough  $x > 0$ ,  $l(x) \in C^+$ . Observe we have:

$$(23) \quad (\bar{a}, f) = (\alpha, e) < 0.$$

Since  $f \in \text{cone}(\Pi)$ , by Equation (23), for large enough  $x$ ,  $l(x)$  must exit  $C^+$ . Then  $l(x)$  must meet another codimension 1 face of  $C^+$ , defined by  $\beta \in \Pi$ , at  $u' \in C^+$  and that  $(\beta, \bar{a}) < 0$ . Define  $w' = ws_\alpha s_\beta$ , by Proposition 2.6,  $w' \in W_{\text{reg}}$ .

Let  $e' \in \mathfrak{h}^*$  be such that  $(1 - w')e' = f$ . Let  $c = \frac{(\beta, \beta)}{2(\beta, \bar{a})}$ , a simple computation shows that  $(1 - w'^{-1})(c\bar{a}) = \beta$ . We compute:

$$(24) \quad \begin{aligned} (\beta, e') &= \left( (1 - w'^{-1})(c\bar{a}), e' \right) \\ &= c(\bar{a}, f) > 0. \end{aligned}$$

To see  $c(\bar{a}, f) > 0$ , we have already observed that  $c < 0$ , then use Equation (23).

Hence, by Equation (24), for  $y > 0$  small enough  $u' + ye' \in C^+$  and  $(1 - w')(u' + ye') = v_{\Delta+y}$ , contrary to the definition of  $\Delta$ .  $\square$

**3. Affine Kac-Moody Algebras** In this section we characterize the closed top dimensional tiles in affine Kac-Moody algebras. We prove the following theorem:

**THEOREM 3.1.**  $\text{Cone}(\Pi) \setminus \mathbb{R}_+\delta = \bigcup_{w \in W} \overline{C_w}$ .

In order to prove this theorem we introduce some notation. Let  $\Pi = \{\alpha_0, \dots, \alpha_n\}$  be the underlying root basis. Recall from [3, Chapter 6], by using distinguished elements  $\Lambda$  and  $\delta = \sum_{i=0}^n a_i \alpha_i$ ,  $a_i \in \mathbb{Z}_+$ , we can decompose the  $\mathfrak{h}^*$  for the affine Kac-Moody algebra as:

$$(25) \quad \mathfrak{h}^* = \widehat{\mathfrak{h}}^* \oplus \mathbb{R}\Lambda + \mathbb{R}\delta$$

where  $\widehat{\mathfrak{h}}^* = \text{span}\{\alpha_1, \dots, \alpha_n\}$ . Let  $\widehat{\Pi} := \{\alpha_1, \dots, \alpha_n\}$ , then the pair  $(\widehat{\mathfrak{h}}^*, \widehat{\Pi})$  is the root system of a finite simple Lie algebra. The invariant form restricted to  $\widehat{\mathfrak{h}}^*$  is an inner product. We also have the following relations satisfied by the invariant form:

$$(26) \quad \begin{aligned} (\delta, \alpha_j) &= 0, \quad j = 1, \dots, n \\ (\Lambda, \alpha_j) &= 0, \quad j = 1, \dots, n \\ (\delta, \delta) &= 0 \\ (\Lambda, \Lambda) &= 0 \\ (\delta, \Lambda) &= 1. \end{aligned}$$

Let  $\widehat{W}$  denote the finite Weyl group of root system  $(\widehat{\mathfrak{h}}^*, \widehat{\Pi})$ . Then  $W = \widehat{W} \rtimes \widehat{M}$ , where  $\widehat{M}$  is defined by a  $\widehat{W}$  invariant lattice  $M \subseteq \widehat{\mathfrak{h}}^*$  (see [3, Chapter 6]). For  $\alpha \in M$ , as element in the lattice, we define the group element  $t_\alpha \in \widehat{M}$  as:

$$(27) \quad t_\alpha(x) = x + (\delta, x)\alpha - \left( (x, \alpha) + \frac{1}{2}|\alpha|^2(\delta, x) \right)\delta, \quad x \in \mathfrak{h}^*.$$

The semi-direct product structure is given as  $t_{w(\alpha)} = wt_\alpha w^{-1}$  for  $w \in \widehat{W}$ ,  $t_\alpha \in \widehat{M}$ . Since  $M$  is  $\widehat{W}$  invariant, it follows that every element of the Weyl group  $W$  can be written as  $wt_\alpha$ , with  $w \in \widehat{W}$ ,  $t_\alpha \in \widehat{M}$ . In light of previous notation, here  $\mathfrak{S} = \mathbb{R}\delta$ ,  $\text{span}\{\Pi\} = \widehat{\mathfrak{h}}^* \oplus \mathbb{R}\delta$ . For the purpose of the rest of this section it is more convenient to define  $\widehat{C}^+ := C^+ \cap \widehat{\mathfrak{h}}^* \oplus \mathbb{R}\Lambda$ , a pointed convex polyhedral cone such that  $C_w = (1-w)(\widehat{C}^+)$ .

Since  $\widehat{C}^+$  is a pointed convex polyhedral cone it can be written as the convex hull of its 1 dimensional faces. We claim that each of its one dimensional faces can be written in the form  $\mathbb{R}_+(v + \Lambda)$ ,  $v \in \widehat{\mathfrak{h}}^*$ . If not, suppose a 1 dimensional face takes the form  $\mathbb{R}_+v$ ,  $v \in \widehat{\mathfrak{h}}^*$ . Then there is some  $i \in \{0, \dots, n\}$  such that  $(v, \alpha_i) > 0$  and  $(v, \alpha_j) = 0$ ,  $j \neq i$ . But this contradicts the fact  $(\delta, v) = 0$ . Hence

all codimension 1 faces take the form  $\mathbb{R}_+(v + \Lambda)$ ,  $v \in \widehat{\mathfrak{h}}^*$ . Thus  $\widehat{C}^+$  admits the following description:

$$(28) \quad \widehat{C}^+ = \text{hull}\{\mathbb{R}_+(v_j + \Lambda), j = 0, \dots, n, v_j \in \widehat{\mathfrak{h}}^*\},$$

where  $\text{hull}\{ \}$  denotes convex hull.

PROOF. We first show the interior of  $\text{cone}(\Pi) \setminus \mathbb{R}_+\delta$  is covered by the closed top dimensional tiles. From Proposition 2.7, it suffices to prove the following:

$$(29) \quad \text{If } T \text{ is a line segment in } \text{int}(\text{cone}(\Pi)) \text{ that does not intersect } \mathbb{R}_+\delta, \text{ then } T \text{ only intersects finitely many top dimensional tiles.}$$

We consider the maps  $1 - w : \widehat{\mathfrak{h}}^* \oplus \mathbb{R}\Lambda \rightarrow \widehat{\mathfrak{h}}^* \oplus \mathbb{R}\delta$ ,  $w \in W_{reg}$ . Let  $H_\delta$  denote the  $\delta = 1$  hyperplane in  $\widehat{\mathfrak{h}}^* \oplus \mathbb{R}\delta$ . We will consider intersections of  $C_w$  with  $H_\delta$ . We also project  $T$  onto this hyperplane via:

$$(30) \quad \widetilde{T} = \frac{T}{\pi_\delta(T)}$$

where  $\pi_\delta(x)$  denotes the  $\delta$  component of  $x \in \widehat{\mathfrak{h}}^* \oplus \mathbb{R}\delta$ . Since  $T \in \text{int}(\text{cone}(\Pi))$ ,  $\pi_\delta(T) \neq 0$ . Hence  $\widetilde{T}$  intersects  $C_w \cap H_\delta$  if and only if  $T$  intersects with  $C_w$ . We also note by construction  $\widetilde{T}$  is a compact continuous curve that does not go through the origin of  $H_\delta$ , where we identify  $H_\delta$  with  $\widehat{\mathfrak{h}}^*$ . To describe of  $C_w \cap H_\delta$ , recall from Equation (28):

$$(31) \quad \widehat{C}^+ = \text{hull}\{\mathbb{R}_+(v_j + \Lambda), j = 0, \dots, n, v_j \in \widehat{\mathfrak{h}}^*\},$$

Then it follows that:

$$(32) \quad C_w = \text{hull}\{\mathbb{R}_+(1 - w_0 t_\alpha)(v_j + \Lambda), j = 0, \dots, n, v_j \in \widehat{\mathfrak{h}}^*\},$$

where  $w = w_0 t_\alpha$ ,  $w_0 \in \widehat{W}$ ,  $t_\alpha \in \widehat{M}$ . Let us then consider:

$$(33) \quad \begin{aligned} (1 - w_0 t_\alpha)(v_j + \Lambda) &= (1 - w_0 t_\alpha)v_j + (1 - w_0 t_\alpha)\Lambda \\ &= (1 - w_0)v_j - w_0(\alpha) + \left( (v_j, \alpha) + \frac{1}{2}|\alpha|^2 \right) \delta \end{aligned}$$

If  $\pi_\delta((1 - w_0 t_\alpha)(v_j + \Lambda)) \neq 0$ , for  $j = 0, \dots, n$ , then  $C_w \cap H_\delta$  are polytopes that are described by:

$$(34) \quad C_w \cap H_\delta = \text{hull} \left\{ \frac{(1 - w_0)v_j - w_0(\alpha)}{\left( (v_j, \alpha) + \frac{1}{2}|\alpha|^2 \right)}, v_j \in \widehat{\mathfrak{h}}^*, j = 0, \dots, n \right\} \subset H_\delta \cong \widehat{\mathfrak{h}}^*.$$

Note because of the quadratic term of  $|\alpha|$  in  $\pi_\delta((1-w_0t_\alpha)(v_j+\Lambda))$ , for large enough  $|\alpha|$  the term  $\pi_\delta((1-w_0t_\alpha)(v_j+\Lambda))$  will be nonzero. Hence there are only finitely many  $C_w$  where  $C_w \cap H_\delta$  is not a polytope. As we have identified  $H_\delta$  with  $\widehat{\mathfrak{h}}^*$ ,  $H_\delta$  inherits an inner product from  $\widehat{\mathfrak{h}}^*$ . The points of the form  $\frac{(1-w_0)v_j-w_0(\alpha)}{((v_j,\alpha)+\frac{1}{2}|\alpha|^2)}$  in  $H_\delta$  that describes the convex hull of  $C_w \cap H_\delta$  in (34) have distance from the origin proportional to  $\frac{1}{|\alpha|}$ . This means as  $|\alpha|$  becomes arbitrarily large, these polytopes converge to the origin.

To complete our proof of (29) suppose for sake of contradiction  $T$  passes through infinite number of top dimensional tiles  $\{C_{w_1}, C_{w_2}, \dots\}$ , then for large enough  $N$ ,  $\widetilde{T}$  passes through infinite number of polytopes  $\{C_{w_{N+1}} \cap H_\delta, C_{w_{N+2}} \cap H_\delta, \dots\}$ . We decompose each  $w_j$  as  $w_j = w_{0_j} t_{\alpha_j}$ . Since there are infinite number of polytopes and  $\widehat{W}$  is a finite group, the  $\alpha_j$  in the lattice  $M$  have arbitrary large norms, hence a subsequence of these polytopes must converge to 0 in  $H_\delta$ . Since  $\widetilde{T}$  is compact continuous curve that intersects  $\{C_{w_{N+1}} \cap H_\delta, C_{w_{N+2}} \cap H_\delta, \dots\}$ , it must go through the origin of  $H_\delta$ , contrary to our assumptions. This concludes the proof of (29).

Now that we have shown the tiles cover the interior of  $\text{cone}(\Pi) \setminus \mathbb{R}_+\delta$ , we consider the boundary. Suppose  $x \in \partial(\text{cone}(\Pi))$  and  $x \neq 0$  (the case  $x = 0$  is covered by  $w = 1$ ). Since the interior of  $\text{cone}(\Pi) \setminus \mathbb{R}_+\delta$  is covered by closed top dimensional tiles, there exists sequence:

$$(35) \quad \{x_n \in \overline{C_{w_n}}, w_n \in W_{reg}\}$$

such that  $x_n \rightarrow x$ . The proof of (29) then demonstrates that the collection  $\{C_{w_n}\}$  can only contain finitely many distinct tiles, for otherwise there would exist a infinite subsequence of  $\{C_{w_n}\}$  of top dimensional tiles whose intersection with  $H_\delta$  converges to origin of  $H_\delta$ , and the fact  $x_n \rightarrow x$  would imply  $x = 0$  in  $\mathfrak{h}^* \oplus \mathbb{R}\delta$ , contrary to our assumptions. Using once again  $x_n \rightarrow x$ , we see  $x$  must be on the closure of a top dimensional tile.

We have shown that:

$$(36) \quad \text{cone}(\Pi) \setminus \mathbb{R}_+\delta \subset \bigcup_{w \in W} \overline{C_w} \subset \text{cone}(\Pi).$$

To conclude our proof of Theorem 3.1 it suffices to show that  $\mathbb{R}_+\delta \not\subset \overline{C_w}$  for any  $w \in W$ . This is accomplished by the next proposition.  $\square$

**PROPOSITION 3.2.** *Let  $\lambda \in \overline{C^+}$ ,  $w \in W$ . Then  $(1-w)\lambda \neq k\delta$  unless  $k = 0$ .*

**PROOF.** We induct on  $l(w)$ , where  $l(w)$  denotes the length of  $w$  (see [3, Chapter 3]).

Recall that the tiles  $C_w$  lie on  $Q := \text{span}\{\alpha_0, \dots, \alpha_n\}$ . Observe for affine Kac-Moody root systems the only vectors on  $Q$  whose magnitude are zero are multiples of  $\delta$ . The case  $l(w) = 1$  is clear, now suppose our hypothesis holds for

all  $w \in W$  such that  $l(w) \leq N$ . Consider  $w' = ws_{\alpha_i}$  such that  $l(w) = N$  and  $l(w') = N + 1$ . For  $\lambda \in \overline{C^+}$ , consider:

$$\begin{aligned}
 (37) \quad (1 - w')\lambda &= \lambda - ws_{\alpha_i}(\lambda) \\
 &= \lambda - w\left(\lambda - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i\right) \\
 &= (1 - w)\lambda + \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}w(\alpha_i).
 \end{aligned}$$

Now for sake of contradiction suppose  $(1 - w')\lambda = k\delta$ ,  $k \neq 0$ , then we have:

$$(38) \quad \left((1 - w')\lambda, (1 - w')\lambda\right) = 0.$$

By the invariance of the bilinear form, Equation (38) is equivalent to:

$$\begin{aligned}
 (39) \quad \left((1 - w')\lambda, \lambda\right) &= \left((1 - w)\lambda, \lambda\right) + \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}\left(w(\alpha_i), \lambda\right) \\
 &= 0
 \end{aligned}$$

Since  $w'$  is reduced,  $w(\alpha_i) \in \text{cone}(\Pi)$  (see [3, Chapter 3]). Since  $\lambda \in \overline{C^+}$ , every term on the right hand side of Equation (39) is  $\geq 0$ , hence they are all 0. This implies  $\left((1 - w)\lambda, (1 - w)\lambda\right) = 2\left((1 - w)\lambda, \lambda\right) = 0$ . By induction hypothesis,  $(1 - w)(\lambda) = 0$ , which combined with the last line of (37), implies that  $(1 - w')\lambda = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}w(\alpha_i)$ , contradicting our initial assumption that  $(1 - w')\lambda = k\delta$ ,  $k \neq 0$ .  $\square$

**Remark** Observe that the proof of Proposition 3.2 actually gives us something slightly stronger: if  $\lambda \in \overline{C^+}$ , then  $\left((1 - w)\lambda, (1 - w)\lambda\right) > 0$ .

**Example 2** We show the computation for two examples, the affine Kac-Moody algebras  $A_2^{(1)}$  and  $C_2^{(1)}$ . We plot the intersection of all tiles with the hyperplane  $H_\delta$ , i.e.  $\coprod_{w \in W} C_w \cap H_\delta$ . The polytopes are top dimensional tiles, the lines are codimension 1 tiles and the dots represent codimensional 2 tiles. In this note we only concerned ourselves with closed top dimensional tiles. It is an open question whether or not we can replace the right hand side of Theorem 3.1 with  $\coprod_{w \in W} C_w$ .

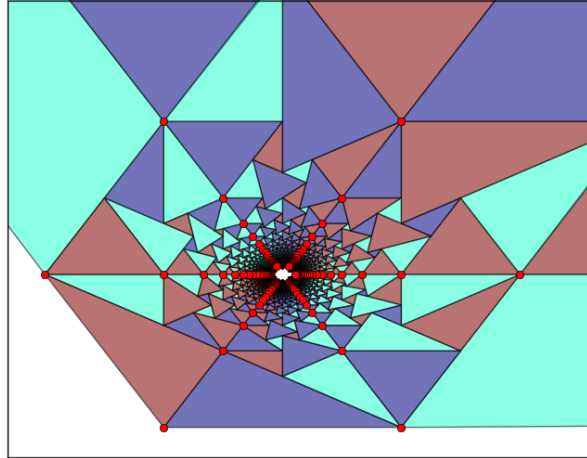


Figure 2: Tiling of the root system  $A_2^{(1)}$

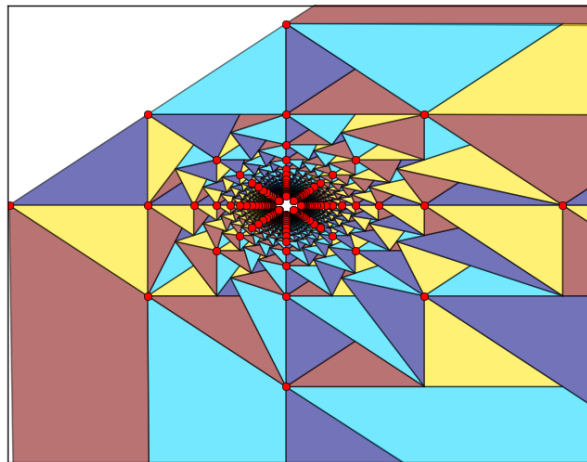


Figure 3: Tiling of the root system  $C_2^{(1)}$

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