C. R. Math. Rep. Acad. Sci. Canada Vol. **39** (2) 2017, pp. 67–76

Further Remarks on Rational Albime Triangles

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Dedicated to Professor Richard K. Guy on his 100th birthday.

Presented by Pierre Milman, FRSC

Abstract. In this note we present further number theoretic properties of the rational albime triangles, in particular, the distribution of acute vs. obtuse rational albime triangles. The notion of albime triangle is extended to include the case of external angle bisector. The proportion of internal vs. external rational albime triangles is also computed.

Résumé. Dans cette note, nous présentons des propriétés supplémentaires (concernant la théorie des nombres) des triangles rationnels 'albimes'; en particulier, la distribution des triangles rationnels albimes aigus contre obtus. La notion de triangle albime est développé pour comprendre le cas d'extérieur bissectrice. On calcule aussi la proportion des triangles rationnels albimes internes contre externes.

1. **Introduction** It was well known to the Greeks that the three altitudes of a triangle are concurrent, and that the same is true of the angle bisectors and the medians. However, the altitude from one vertex, the angle bisector and the median from the other two are rarely concurrent. An interesting discussion of such topics is given in [2]. The concurrence is even rarer if the side lengths of such triangles are assumed to be whole numbers, or equivalently, rationals. The existence of such triangles has been asked for and forgotten many times over the last eighty years and has a rich history (cf. [1]). It is only relatively recently when R. Guy [6] produced dozens of examples of such triangles with side lengths whole numbers. In fact, he gave an algorithm to generate them all and conjectured that his algorithm would produce infinitely many of them. His conjecture on their infinitude was proved in [1].

As a side remark, a solution using elliptic curves to the problem that Hoyt proposed in the Monthly (cf. [1]) was found by Charles Toll. However, the Monthly did not publish it at the time because, as it informed Toll, one of its editors (i.e. Richard Guy) was preparing a paper on the subject.

As an appropriate tribute to Richard Guy on his 100th birthday we further explore the issue, suggested by Guy, of taking the angle bisector to be external.

Received by the editors on July 26, 2015; revised November 14, 2016.
AMS Subject Classification: Primary: 11G05; secondary: 11Z05.
Keywords: Elliptic curves, rational albime triangles, Ceva's theorem, primitive albime triplets.

See Figures 1 and 4 for internal vs. external case, the bisector being of the angle of the triangle at A. In these figures, M is the midpoint of the side AC, so BM is the median from B, CF is the altitude from C, so $\angle CFB$ is a right angle. Finally, the line AO is the bisector of angle A, in Figure 1 internally, whereas in Figure 4, externally.

2. Albime Triangles

The study of albime triangles with side lengths rational provides a beautiful illustration of a blend of algebra, arithmetic, geometry, and some distribution results. Although the name albime had not been coined yet, the subject inspired by Guy's paper is treated in [2].

An *albime triangle* is a triangle with the property that for a certain labeling A, B, C of its vertices, the altitude from C, the internal angle bisector at A and the median from B are concurrent. For a subfield K of $\mathbb{R}$ if the side lengths $a = |BC|$, $b = |AC|$, $c = |BA|$ of the albime triangle ΔABC are in K, we call it a K-*albime triangle*. In particular, if $K = \mathbb{Q}$, it is a *rational albime triangle*.

An albime triangle is uniquely determined by the side AC and the angle A, because if the altitude from C and the angle bisector at A concur at O and M is the midpoint of AC, then the line MO meets the line AE at a unique point B (see Figure 1) to make an albime triangle with vertices A, B, and C

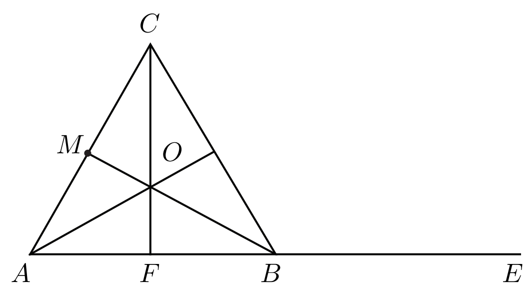

Figure 1: Albime triangle with side AC and the angle at A already given.

Albime is a property of the equivalence class under the similarity of triangles. Among each equivalence class of rational albime triangles there is a unique one with side lengths a, b, c in $\mathbb{N}$ and having no common factor larger than 1. We call such a triplet (a, b, c) a *primitive albime triplet*. The trivial primitive triplet is $(1, 1, 1)$. Among the first non-trivial primitive albime triplets are $(12, 13, 15)$, $(35, 277, 308)$, $(26598, 26447, 3193)$, $(610584, 587783, 4832143)$, etc.

Determination of (primitive) albime triplets is a non-trivial number theoretic problem. Richard Guy [6] found an algorithm to produce all such triplets and predicted that there are infinitely many of them, which subsequently was proved to be true in [1]. There are acute and obtuse albime triangles. Guy also considers *external albime triangles* (though not using this terminology) where the angle bisector is external. In this note we compute the ratios, in a certain sense, of various kinds (acute vs. obtuse, and internal vs. external) of equivalence classes of rational albime triangles.

3. Albime Triangles and Elliptic Curves It is known ([1], [6]) that the points $P = (x, y)$ with real coordinates $y > 0$ and $0 < x < 2$ on Guy's elliptic curve E defined by

$$y^2 = x^3 - 4x + 4 \tag{1}$$

correspond bijectively to the equivalence classes, with respect to similarity of triangles, of albime triangles. More precisely, $P = (x, y)$ in this way corresponds to the class containing the triangle with side lengths $(a = y, b = 2 - x, c = x)$.

Let $E(K)$ denote the group of points $P = (x, y)$ on (1) with x, y in K together with the point O at infinity. Then the equivalence classes containing K-albime triangles correspond bijectively to the points P in $E(K)$ with y-coordinate $y(P) > 0$ and x-coordinate $x(P)$ in the open interval $(0, 2)$.

The case of rational albime triangles is of special interest from the number theoretic point of view. It is known (see [3]) that the group $E(\mathbb{Q})$ is an infinite cyclic group generated by the point $P = (2, 2)$. It was stated in [6] and proved in [1] that about 36% of the points in $E(\mathbb{Q})$ correspond to the equivalence classes of rational albime triangles in the sense that

$$\lim_{N \to 0} \frac{\#\{n \in \mathbb{Z} \mid |n| \leq N \text{ and } 0 < x(nP) < 2\}}{2N + 1} = .3612\ldots.$$

4. Acute or Obtuse? The angle at vertex C, which we always draw the altitude from, can take any value between zero and 180°. When we say an albime triangle is acute, obtuse or right triangle, we mean that the angle C is acute, obtuse or right, respectively.

The middle point $c = 1$ of the open interval $(0, 2)$ corresponds to the primitive albime triplet $(1, 1, 1)$ representing the equivalence class of equilateral triangles, which of course are albime. The class corresponds to the point $(1, 1) = -4P$ where $P = (2, 2)$ generates the infinite cyclic group $E(\mathbb{Q})$. A quick look at the two rational albime triangles, one acute, the other obtuse, that correspond to the points $-10P$ and $13P$ of $E(\mathbb{Q})$ (see Figures 2 and 3) suggests that as c approaches 2, the albime triangles look more and more like toothpicks, whereas as c approaches 0, they look more and more like a slivery wedge. This observation suggests there is a unique value c_0 of c in the open interval $(0, 2)$ such that the point (c, a) in $E(\mathbb{Q})$ corresponds to an obtuse albime triangle if and only if c is in the open interval $(c_0, 2)$. We prove herein that this is true. Throughout we label the vertices A, B, C of our albime triangle as in its definition. Recall that (e.g. see [1]) the correspondence between the points (c, a) in $E(K)$ and the equivalence classes of K-albime triangles is given by

$$(c, a) \leftrightarrow (\sqrt{c^3 - 4c + 4}, 2 - c, c),$$

i.e. the point (c, a) in $E(K)$ with $a > 0$ and $0 < c < 2$ corresponds to the K-albime triangle in its equivalence class with side lengths $a = \sqrt{c^3 - 4c + 4}$, $b = 2 - c$ and c.

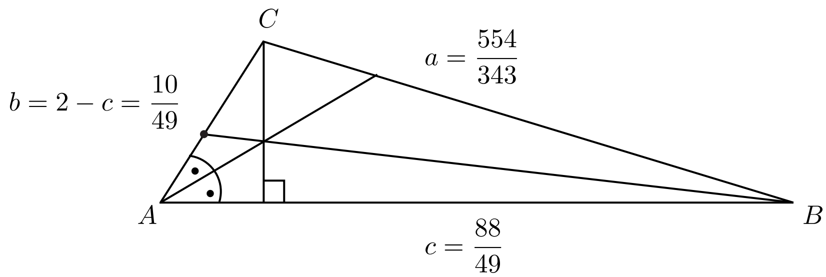

Figure 2: Rational albime triangle corresponding to $-10P$.

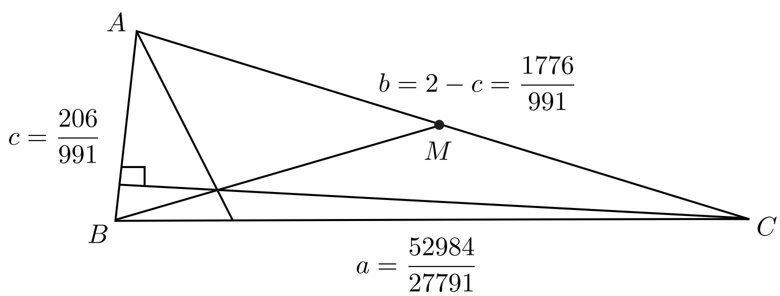

Figure 3: Rational albime triangle corresponding to $13P$.

PROPOSITION 1. *In any albime triangle, the angles A and B are acute.*

PROOF. If to the contrary A is obtuse, then $\cos A < 0$, so by the Law of Cosines,

$$b^2 + c^2 < a^2. \tag{2}$$

Since the triangle is albime, we may take c in the open interval $(0, 2)$, $b = 2 - c$ and $a^2 = c^3 - 4c + 4$. Then (2) implies that $c^2(2 - c) < 0$, which is impossible. The proof that B is also acute is similar. □

Proposition 2. *An albime triangle with the side length c in the open interval $(0,2)$ is obtuse if and only if $\sqrt{5}-1<c<2$.*

Proof. By Proposition 1, the angle C is obtuse, hence $b^2+c^2<a^2$. Again, substituting the values of a, b in terms of c for albime triangles,

$$c^2+2c-4>0. \tag{3}$$

By looking at the graph of the parabola $s=t^2+2t-4$, (3) holds if and only if $c>\sqrt{5}-1$. □

Remark 1. The unique value $\sqrt{5}-1$ of c at which the albime triangle is a right triangle (clearly not a rational albime triangle) corresponds to the point $Q=(\sqrt{5}-1,\sqrt{\sqrt{5}-2})$ on E. The smallest field containing the coordinates of Q is $K=\mathbb{Q}(\sqrt{5},\sqrt{\sqrt{5}-2})$, an extension of $\mathbb{Q}$ of degree 4. It may be interesting to know that $E(K)$ is a torsion-free group of rank 2 with generators $P_1=P=(2,2)$ and $P_2=(\alpha^3+5\alpha+2,3\alpha^3+\alpha^2+11\alpha+3)$, α being a root of the polynomial t^4+4t-1. The point Q can be expressed in terms of the generators P_1 and P_2 as

$$Q=\left(\sqrt{5}-1,\sqrt{\sqrt{5}-2)}\right)=2P_1-P_2.$$

It is a non-trivial fact (see [5]) that extending $\mathbb{Q}$ to $K=\mathbb{Q}(\sqrt{5})$ does not produce new K-albime triangles.

5. External Albime Triangles

The essential idea for the following definition already occurs in Guy's paper [6]. We prove that his prediction of the frequency $2/11$ with which such triangles are produced by the points nP with n in $\mathbb{Z}$ is not far from the truth.

Definition . A triangle with certain labeling A, B, C of its vertices is *externally albime* if the altitude from C, the external bisector of angle A and the median from C are concurrent (see Figure 4).

The following is an analog of (Theorem 3.2 [1]).

Theorem 1. *For a subfield K of $\mathbb{R}$, the equivalence classes containing external K-albime triangles correspond bijectively to the points (c,a) on (1) with $a>0$ and $-2<c<0$. A correspondence is given by*

$P=(c,a)\leftrightarrow$ *the external albime triangle with side lengths*

$\sqrt{c^3-4c+4}, 2-c$ *and* $-c$.

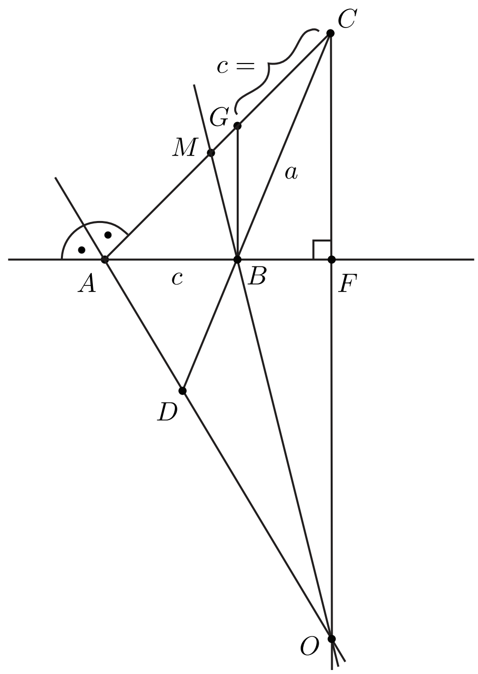

Figure 4: An external albime triangle.

Proof. Ceva's Theorem and Euclid's Angle Bisector Theorem both hold for external angle bisectors as well. We need the following modification of the proof of (Theorem 3.2 [1]). Let O be the point of concurrence of the given albime triangle and BG the line parallel to the altitude from C (see Figure 4). By the Angle Bisector Theorem,

$$\frac{|BD|}{|DC|} = \frac{c}{b}. \tag{4}$$

Since M is the midpoint, (4) and Ceva's Theorem imply that

$$\frac{|AF|}{|FB|} = \frac{b}{c}. \tag{5}$$

By the similarity of triangles,

$$\frac{|GC|}{b} = \frac{|BF|}{|AF|}. \tag{6}$$

Equations (5) and (6) imply that $|GC| = c$. Now scale the given albime triangle by a suitable element of K so that $|AG| = 2$, i.e. $|AC| = 2 + c$. If $|BF| =$

d, computing the altitude $|CF|$ in two different ways, using the Pythagorean Theorem, we obtain

$$d = \frac{4c + 4 - a^2}{2c}. \tag{7}$$

Rewriting equation (6) as

$$\frac{c + d}{d} = \frac{b}{c}$$

also gives $d = c^2/2$. Thus

$$\frac{4c + 4 - a^2}{2c} = \frac{c^2}{2},$$

which leads to the points $(-c, \pm a)$ on Guy's elliptic curve defined by (1). For our purpose, we take $P = (-c, a)$.

The converse is a triviality. □

6. **Distribution of Different Kinds of Albime Triangles** We now show that among roughly 36% of the points in $E(\mathbb{Q})$ that correspond to (internal) albime triangles, about 22.45% correspond to acute, while the remaining 13.67% to the obtuse albime triangles. The percentage of the points in $E(\mathbb{Q})$ that correspond to external albime triangles is 19.4%. Thus well over half, namely about 55%, of the points in $E(\mathbb{Q})$ correspond to internal and external rational albime triangles.

To compute these percentages, we use two facts. The first is the following well-known result due independently to various people [9], [8], [4]. It is a theorem whose roots go back at least to Kronecker, and is a corollary to what is now usually called Weyl's criterion on uniform distribution. A readable proof can be found in [7] where it is called Kronecker's theorem.

THEOREM 2. *If z is not an element of finite order of the multiplicative group $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, then the cyclic group generated by z is uniformly distributed in $\mathbb{T}$.*

The second idea is the following result.

THEOREM 3. *There is a bicontinuous group isomorphism Ψ between the additive group $E(\mathbb{R})$ and the multiplicative group $\mathbb{T}$.*

This is well known. For a constructive proof, see [1]. In fact, we recall that an explicit isomorphism (needed for our computations) can be defined as follows.

Let

$$\alpha = -2.382975767\ldots$$

be the only real root of the polynomial

$$f(t) = t^3 - 4t + 4.$$

For a point $P = (x, y)$, let $x = x(P)$ and $y = y(P)$ be the x and y-coordinates of P. Thus $x(nP)$ is the x-coordinate of the point $Q = \underbrace{P + \cdots + P}_{n\text{-times}}$, the addition being the group law on the elliptic curve E.

First define the map ϕ from $E(\mathbb{R})$ to the unit interval by

$$\phi(P) = \frac{1}{2}\left[\int_{x(P)}^{\infty} \frac{dt}{\sqrt{f(t)}} \bigg/ \int_{\alpha}^{\infty} \frac{dt}{\sqrt{f(t)}}\right]$$

if $y(P) \geq 0$, and $\phi(P) = 1 - \phi(-P)$, if $y(P) \leq 0$. Finally let $\phi(O) = 0$ and put

$$\Psi(P) = e^{2\pi i \phi(P)}.$$

Recall that $P = (2, 2)$ is the designated generator of the group $E(\mathbb{Q})$, whereas $Q = (\sqrt{5} - 1, \sqrt{\sqrt{5} - 2})$ is the point on $E(\mathbb{Q}(\sqrt{5}, \sqrt{\sqrt{5} - 2}))$ that corresponds to the right albime triangle. The other relevant points for computing the aforementioned percentiles are (see Figure 5)

$$2P = (0, 2) \quad \text{and} \quad -3P = (-2, 2).$$

Since $t \to e^{2\pi i t}$ maps subinterval of the closed interval $[0, 1]$ of equal lengths to arcs of $\mathbb{T}$ of equal lengths, by Theorem 2, these percentiles come from the following limits:

$$\lim_{N\to\infty} \frac{\#\{n \in \mathbb{Z} \mid |n| \leq N,\ 0 < x(nP) < \sqrt{5} - 1\}}{2N + 1}$$

$$= 2(\phi(2P) - \phi(Q)) = 0.224486\ldots,$$

$$\lim_{N\to\infty} \frac{\#\{n \in \mathbb{Z} \mid |n| \leq N,\ \sqrt{5} - 1 < x(nP) < 2\}}{2N + 1}$$

$$= 2(\phi(Q) - \phi(P)) = 0.136722\ldots,$$

and

$$\lim_{N\to\infty} \frac{\#\{n \in \mathbb{Z} \mid |n| \leq N,\ -2 < x(nP) < 0\}}{2N + 1}$$

$$= 2(\phi(-3P) - \phi(2P)) = 0.193960\ldots,$$

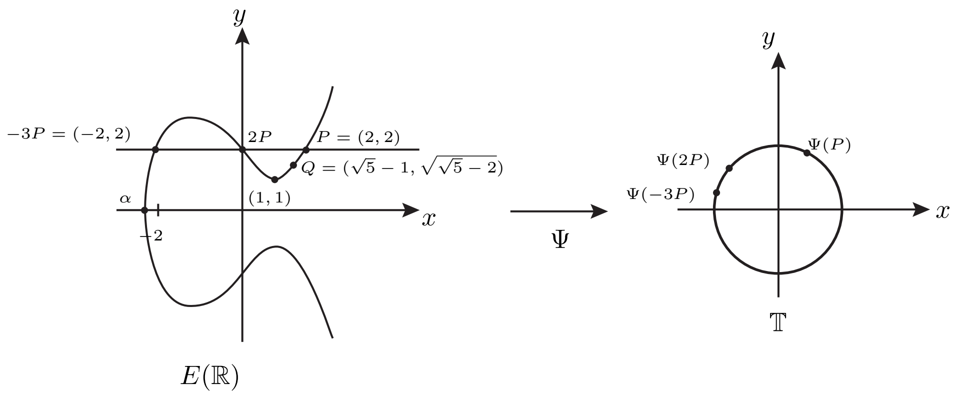

Figure 5: $E(\mathbb{R}) \cong \mathbb{T}$

To conclude (see Figure 6), of all points in $E(\mathbb{Q})$:

$\approx$ 44.4832% yield no albime triangle;
$\approx$ 19.3960% yield rational albime triangles with external bisector;
$\approx$ 22.4487% yield acute rational albime triangles;
$\approx$ 13.6721% yield obtuse rational albime triangles.

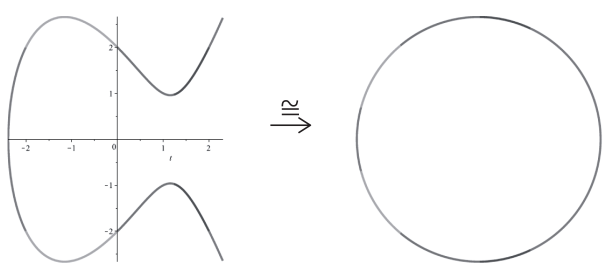

Figure 6

Acknowledgement. The authors would like to thank the referee for his helpful suggestions to make the exposition clearer. The first author would like to thank the third author and the University of Groningen for their hospitality during the preparation of the first version of this article.

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