

# CAUCHY PROBLEM ON TWO CHARACTERISTIC HYPERSURFACES FOR THE EINSTEIN-VLASOV SCALAR FIELD EQUATIONS IN TEMPORAL GAUGE

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**ABSTRACT.** In this paper, we consider the initial value problem for the Einstein-Vlasov scalar field equations in temporal gauge, where the initial data are prescribed on two characteristic smooth intersecting hypersurfaces. From a suitable choice of some free data, the initial data constraints's problem is solved globally, then the evolution problem relative to the deduced initial data is solved locally in time.

**RÉSUMÉ.** Dans cet article, on considère le problème de Cauchy pour les équations d'Einstein-Vlasov-Champ scalaire en jauge temporelle, dans le cas où les données initiales sont prescrites sur deux hypersurfaces caractéristiques régulières sécantes. A partir d'un choix judicieux de certaines données indépendantes, le problème des contraintes initiales est globalement résolu, et ensuite le problème de l'évolution relatif aux données initiales déduites est résolu localement dans le temps.

**1. Introduction** In this paper, we prove a local (in time) well-posedness result for the characteristic Cauchy problem for the Einstein-Vlasov scalar field equations in temporal gauge, when the data are assigned on two characteristic intersecting smooth hypersurfaces. Indeed, in General Relativity, there are basically two types of Cauchy problem: the ordinary spacelike Cauchy problem for the Einstein equations, and the characteristic initial value problem for these same equations. In the first case, the system of constraints's equations is standard, i.e., the Hamiltonian and momentum constraints ([2], Chap. 7; [3]); they depend only on the nature of the Einstein equations, and are independent of the choice of the gauge and of the corresponding evolution system. In the case of the characteristic Cauchy problem, the set of constraints's equations includes the standard constraints and additional constraints induced by the choice of the gauge, the evolution system considered, the free data, and obviously the form of the stress-energy momentum tensor of the matter considered. The gauge commonly used for the characteristic Cauchy problem is the harmonic gauge; see [14] where the Einstein equations in vacuum or perfect fluid are considered, with data assigned

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on two null intersecting smooth hypersurfaces, [7],[8],[9] for the study of the Einstein-Yang-Mills equations with data prescribed on two intersecting smooth hypersurfaces, [10] where various aspects of the characteristic Cauchy problem in general relativity are reviewed, [6] for the analysis of the Cauchy problem on a characteristic cone for the Einstein equations, and [5] where kinematic matter is considered for the Cauchy problem on a characteristic cone, using the generalized wave map gauge, in the important case of astrophysical studies. Another gauge used recently by other authors is the “Double null foliation gauge” introduced for the study of the Cauchy problem for the Einstein equations in vacuum [1],[11]. In the presence of the Vlasov field, the hierarchical method of resolution of the constraints developed by A.D. Rendall [14] in the case of the harmonic gauge is less suitable; this is due to some difficulties caused by the presence of all the components of the metric in each component of the stress-energy momentum tensor generated by the Vlasov’s matter. Such difficulties are mentioned in [5],[12],[15]. Furthermore, C. Tadmon in [15] attempted to solve this problem in harmonic coordinates, but this author was forced to impose unnatural restrictive conditions (of integral type) for the initial density of the particles on the initial hypersurfaces. To tackle these difficulties, we opted for the temporal gauge. Following Y. Choquet-Bruhat [3],[2], we chose the evolution system; and then highlighted the system of constraints which are of two kinds, the usual Hamiltonian and momentum constraints (3.5), and other constraints (3.6) due to the condition of temporal gauge, all described in coordinates  $(y^\delta)$  (3.1), adapted to the geometry of the initial hypersurfaces. The constraints  $\tilde{\mathcal{C}}_{ab} - \frac{\tilde{g}^{cd}\tilde{\mathcal{C}}_{cd}}{n-1}\tilde{g}_{ab} = 0$ ,  $(a, b = 2, \dots, n)$ , extracted from (3.6) have their analogue in the “Double null foliation gauge” [1]. Theorem 1 of the present paper resumes in the setting of temporal gauge the entire set of the constraints to solve from given free data, in order to prescribe the full set of initial data for the evolution system considered, while Theorem 2 deals with the resolution of these constraints. For a suitable choice of free data, the system of constraints (3.5)–(3.6) is solved hierarchically. The last part of the article is devoted to the existence theorem for the Einstein-Vlasov scalar field equations. For the sake of simplicity, we have considered  $C^\infty$  initial data.

**2. Geometric Setting and Formulation of the Problem** Let  $(x^\alpha) = (x^0, x^1, x^a)$ ,  $(\alpha = 0, 1, \dots, n; a = 2, \dots, n)$ , denote the global canonical set of coordinates of  $\mathbb{R}^{n+1} = \mathbb{R}^2 \times \mathbb{R}^{n-1}$ ,  $(n \geq 3)$ .  $B$  is a compact domain of  $\mathbb{R}^{n-1}$ ,  $Y := \{(x^\alpha) \in \mathbb{R}^{n+1}, x^0 - |x^1| \geq 0, (x^a) \in B\}$ ,  $\mathcal{I}^0 = \{(x^\alpha) \in Y : x^0 + x^1 = 0\}$ ,  $\mathcal{I}^1 = \{(x^\alpha) \in Y : x^0 - x^1 = 0\}$ . One considers in  $\hat{Y} := Y \times \mathbb{R}^n$  the Cauchy problem for the Einstein-Vlasov scalar field system when the initial hypersurfaces  $\mathcal{I}^0$  and  $\mathcal{I}^1$  are null w.r.t. the prescribed initial data. This system of unknown functions  $(g, \rho, \Phi)$  reads

$$(2.1) \quad H_g : G_{\mu\nu} \equiv R_{\mu\nu} - 2^{-1}g_{\mu\nu}R = T_{\mu\nu},$$

$$(2.2) \quad H_\rho : p^\alpha \frac{\partial \rho}{\partial x^\alpha} - \Gamma_{\mu\nu}^i p^\mu p^\nu \frac{\partial \rho}{\partial p^i} = 0,$$

$$(2.3) \quad H_\Phi : \square\Phi = V'(\Phi).$$

The Einstein equations  $H_g$  describe the gravitational potential  $g$ , while the Vlasov equation gives a statistical description of a collection of particles of rest mass  $\mathbf{m}$  and density  $\rho \equiv \rho(x, p^0, p^i)$ , which move towards the future ( $p^0 > 0$ ) in their mass shell  $\mathbb{P} := \{(x, p^\mu) \in Y \times \mathbb{R}^{n+1} / g_{\mu\nu} p^\mu p^\nu = -\mathbf{m}^2, p^0 > 0\}$ . The wave equation  $H_\Phi$  for the matter field  $\Phi$  (of potential  $V$ ) expresses the divergence free of the stress-energy momentum tensor of matter. The terms  $R_{\mu\nu}$ ,  $R$  and  $G_{\mu\nu}$  designate respectively the components of the Ricci tensor, the scalar curvature, and the components of the Einstein tensor  $G$ , relative to the searched metric  $g$ , while the  $T_{\mu\nu}$  are the components of the stress-energy momentum tensor of matter. The  $\Gamma_{\mu\nu}^\lambda$  are the Christoffel symbols of  $g$  and the  $p^\lambda$  stand as the components of the momentum of the particles with respect to the basis  $(\frac{\partial}{\partial x^\alpha})$  of the fiber  $\mathbb{P}_x := \{(p^\alpha) \in \mathbb{R}^{n+1} / g_{\mu\nu}(x) p^\mu p^\nu = -\mathbf{m}^2, p^0 > 0\}$  of  $\mathbb{P}$ . The  $T_{\mu\nu}$  are given by

$$(2.4) \quad T_{\alpha\beta} = \partial_\alpha \Phi \partial_\beta \Phi - \frac{1}{2} g_{\alpha\beta} (g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + V(\Phi)) \\ - \int_{\{(p,p)=-\mathbf{m}^2\}} \frac{\rho(x^\nu, p^\mu) p_\alpha p_\beta \sqrt{|g|}}{p^0} dp^1 \dots dp^n.$$

In this setting, we choose the temporal gauge [3], contrary to the usual harmonic gauge [3], [5], [10],[14], [15]; i.e., we choose a zero shift and “densitize” the lapse, requiring the time to be in wave gauge:  $g_{0i} = 0$ ,  $\Gamma^0 \equiv g^{\lambda\delta} \Gamma_{\lambda\delta}^0 = 0$ ,  $i = 1, \dots, n$ ;  $\lambda, \delta = 0, 1, \dots, n$ . The metric has then the form

$$(2.5) \quad g = -\tau^2 (dx^0)^2 + \bar{g}_{ij} dx^i dx^j.$$

One denotes by  $\bar{\nabla}$  the connection with respect to the induced metric  $\bar{g}$  on  $\Lambda_t : x^0 = t$ .

Setting  $\Lambda_{\mu\nu} = T_{\mu\nu} + \frac{g^{\lambda\delta} T_{\lambda\delta}}{1-n} g_{\mu\nu}$ , and following Choquet Bruhat (see [3],[2]), one chooses as evolution system attached to  $(H_g, H_\Phi, H_\rho)$ , the system  $(H_{\bar{g}}, H_\Phi, H_\rho)$  where  $H_{\bar{g}}$  is

$$(2.6) \quad H_{\bar{g}} : \partial_0 R_{ij} - \bar{\nabla}_i R_{j0} - \bar{\nabla}_j R_{i0} = \partial_0 \Lambda_{ij} - \bar{\nabla}_i \Lambda_{j0} - \bar{\nabla}_j \Lambda_{i0};$$

and its principal part is  $\square \partial_0 \bar{g}_{ij}$ . The problem considered splits thus into two problems that are the initial data constraints’s problem and the evolution problem for the third order quasi-diagonal Leray hyperbolic system  $(H_{\bar{g}}, H_\Phi, H_\rho)$ . The first consists of studying how to prescribe a large class of initial data  $(\bar{g}_0, k_0, \phi_0, \rho_0)$  on  $(\mathcal{I} = \mathcal{I}^0 \cup \mathcal{I}^1) \times \mathbb{R}^n$  and such that for any solution  $(\bar{g}, \Phi, \rho)$  of the evolution system  $(H_{\bar{g}}, H_\Phi, H_\rho)$  in a neighborhood  $\hat{Y}$  of  $\mathcal{I} \times \mathbb{R}^n$  satisfying  $\bar{g}|_{\mathcal{I}} = \bar{g}_0$ ,  $(\partial_0 \bar{g})|_{\mathcal{I}} = k_0$ ,  $\Phi|_{\mathcal{I}} = \phi_0$ ,  $\rho|_{\mathcal{I}} = \rho_0$ ,  $(g, \Phi, \rho)$  is solution of the Einstein-Vlasov scalar field equations  $(H_g, H_\Phi, H_\rho)$  in  $\hat{Y}$ , where  $g$  is of the form (2.5) with  $\tau^2 = (c(x^i))^2 |\bar{g}|$ , and  $c$  is determined by the prescribed data such that  $\Gamma^0 = 0$  in  $Y$ .

**3. The Characteristic Initial Data Constraints** To describe the full set of constraints, we introduce the coordinates  $(y^\delta)$  defined by

$$(3.1) \quad y^m = x^0 + (-1)^m x^1, \quad y^a = x^a; \quad a = 2, \dots, n; \quad m = 0, 1;$$

and require the assumptions:

$$(3.2) \quad (\mathbf{M})_m : \text{ the vector fields } \frac{\partial}{\partial y^{1-m}} \text{ is tangent to the null geodesics} \\ \text{generating } \mathcal{I}^m, \quad m = 0, 1;$$

clearly sufficient in temporal gauge to carry out the analysis, and which are similar to the affine parametrization conditions of [14],[6]. The components of tensors in coordinates  $(y^\delta)$  are equipped with a tilde “ $\sim$ ”.

The assumptions  $(\mathbf{M})_m$  induce that the trace on  $\mathcal{I}^0$  and  $\mathcal{I}^1$  of the searched metric  $g$  has the specific form

$$(3.3) \quad g|_{\mathcal{I}^m} = \tilde{g}_{01}(dy^0 dy^1 + dy^1 dy^0) + \tilde{g}_{ab} dy^a dy^b; \quad a, b = 2, \dots, n.$$

Correspondingly, the restrictions to  $\mathcal{I}^0$  and  $\mathcal{I}^1$  of the components of the Einstein tensor and the momentum tensor in coordinates  $(y^\delta)$  imply the constraints (3.6) below and the following theorem.

**THEOREM 1.** *Let  $(\bar{g}, \Phi, \rho)$  be any  $C^\infty$  solution of the evolution system  $(H_{\bar{g}}, H_\Phi, H_\rho)$  in a neighborhood  $\mathcal{V}$  of  $\mathcal{I} \times \mathbb{R}^n$ , and let*

$$(3.4) \quad g = -\tau^2(dx^0)^2 + \bar{g}_{ij} dx^i dx^j;$$

and suppose that the temporal gauge condition is satisfied in  $\mathcal{V}$ . One sets  $\tilde{\mathfrak{C}}_{\mu\nu} = \tilde{G}_{\mu\nu} - \tilde{T}_{\mu\nu}$ ,  $\tilde{\mathfrak{C}} = \tilde{g}^{\mu\nu} \tilde{\mathfrak{C}}_{\mu\nu}$  and one assumes that with respect to the metric  $g$ , the hypotheses  $(M_m)$  (3.2) and the relations

$$(3.5) \quad \tilde{\mathfrak{C}}_{\bar{m}\lambda} = 0, \quad \lambda = 0, \dots, n; \quad m = 0, 1; \quad \bar{m} = 1 - m;$$

$$(3.6) \quad \tilde{\mathfrak{C}}_{ab} - \frac{\tilde{g}^{cd} \tilde{\mathfrak{C}}_{cd}}{n-1} \tilde{g}_{ab} = 0; \quad \tilde{\mathfrak{C}}_{mm} - 2 \frac{\tilde{g}_{01}}{n-1} \tilde{\mathfrak{C}} + \tilde{g}_{01} \frac{\partial(\tilde{\Gamma}^0 + \tilde{\Gamma}^1)}{\partial y^m} = 0; \\ a, b, c, d = 2, \dots, n;$$

are satisfied on  $(\mathcal{I}^m), m = 0, 1$ ; then  $(g, \Phi, \rho)$  is a solution of the Einstein-Vlasov scalar field equations  $(H_g, H_\Phi, H_\rho)$  in  $\mathcal{V}$ .

**PROOF.** If  $(\bar{g}, \Phi, \rho)$  is a  $C^\infty$  solution of the evolution system  $(H_{\bar{g}}, H_\Phi, H_\rho)$  and if the relations (3.4), (3.5),(3.6) are satisfied for  $g$  of the form (3.3), then on  $\mathcal{I}^0$

$$(3.7) \quad \mathfrak{C}_{00} = \mathfrak{C}_{01}, \quad \mathfrak{C}_{11} = \mathfrak{C}_{01}, \quad \mathfrak{C}_{1a} = \mathfrak{C}_{0a}, \quad \mathfrak{C}_{ab} = -\frac{\mathfrak{C}_{01}}{g_{11}} g_{ab}, \\ \tilde{\mathfrak{C}} = -\frac{n-1}{g_{11}} \mathfrak{C}_{01}; \quad a, b = 2, \dots, n;$$

furthermore, the divergence free properties of the Einstein tensor ( $G_{\mu\nu}$ ) and the stress energy momentum tensor of matter ( $T_{\mu\nu}$ ) of  $g$  (3.4), imply that on  $\mathcal{I}^0$  one has

$$(3.8) \quad \nabla^\alpha \mathfrak{C}_{\alpha\beta}|_{\mathcal{I}^0} = 0, \quad (\partial_0(R_{ij} - \Lambda_{ij}) - \bar{\nabla}_i \mathfrak{C}_{j0} - \bar{\nabla}_j \mathfrak{C}_{i0})|_{\mathcal{I}^0} = 0.$$

By combining the relations (3.7) and (3.8), one obtains on  $\mathcal{I}^0$  the system

$$(3.9) \quad g^{11} \partial_1 [\mathfrak{C}_{0i}] + L_i([\mathfrak{C}_{0j}]) = 0;$$

where for  $i = 1, \dots, n$ ,  $L_i$  is a linear homogeneous expression in terms of  $[\mathfrak{C}_{0j}] \equiv \mathfrak{C}_{0j}|_{\mathcal{I}^0}$ ,  $j = 1, \dots, n$ . This system has zero data on  $\mathcal{S}$  since the constraints (3.5)–(3.6) are satisfied on  $\mathcal{S}$ , and one deduces that  $[\mathfrak{C}_{0i}] = 0$ , after that,  $\mathfrak{C}_{\mu\nu} = 0$  on  $\mathcal{I}^0$  thanks to the relations (3.7). In the same way it is shown that  $\mathfrak{C}_{\mu\nu} = 0$  on  $\mathcal{I}^1$ . Now, to prove that  $\partial_0 \mathfrak{C}^{\mu\nu} = 0$  on  $\mathcal{I}^0$ , one restricts to  $\mathcal{I}^0$  the following linear homogeneous system satisfied by the  $\mathfrak{C}^{\mu\nu}$  in  $\mathcal{V}$  (see [3], pages [407–414], or [13]),

$$(3.10) \quad \partial_0 \mathfrak{C}^{00} + L^{00}(\mathfrak{C}^{\gamma\alpha}, \partial_i \mathfrak{C}^{i0}) = 0$$

$$(3.11) \quad \partial_0 \mathfrak{C}^{ij} + L^{ij}(\mathfrak{C}^{\gamma\alpha}, \partial_s \mathfrak{C}^{k0}) = 0$$

$$(3.12) \quad \square_g \mathfrak{C}^{0j} + L^{0j}(\mathfrak{C}^{\gamma\alpha}, \partial_s \mathfrak{C}^{\delta\beta}) = 0.$$

By combining these restrictions, the system (3.12) restricted to  $\mathcal{I}^0$  appears as a homogeneous linear system of propagation equations on  $\mathcal{I}^0$  with unknowns the restrictions to  $\mathcal{I}^0$  of  $\partial_0 \mathfrak{C}^{0i}$ ,  $i = 1, \dots, n$ . To show that this system has zero data on  $\mathcal{S}$ , one restricts to  $\mathcal{S}$  the systems (3.10)–(3.11) and first deduces that on  $\mathcal{S}$ ,  $\partial_0 \mathfrak{C}^{00} = 0$ ,  $\partial_0 \mathfrak{C}^{ij} = 0$ ; after that, one uses the restrictions to  $\mathcal{S}$  of the properties  $\nabla_\alpha \mathfrak{C}^{\alpha i} = 0$ ,  $i = 1, \dots, n$  to conclude that  $\partial_0 \mathfrak{C}^{0i} = 0$  on  $\mathcal{S}$ . One deduces that on  $\mathcal{I}^0$ ,  $\partial_0 \mathfrak{C}^{0i} = 0$ ,  $i = 1, \dots, n$  and subsequently  $\partial_0 \mathfrak{C}^{00} = 0$ ,  $\partial_0 \mathfrak{C}^{ij} = 0$  on  $\mathcal{I}^0$ , thanks to the linear system (3.10)–(3.11). We remark that similar reasoning holds on  $\mathcal{I}^1$ . Finally  $\mathfrak{C}^{\mu\nu} = 0$  in  $\mathcal{V}$  thanks to another linear homogeneous hyperbolic system in  $\mathcal{V}$ , derived from (3.10)–(3.12) which has principal part  $\square \partial_0 \mathfrak{C}^{\mu\nu}$  (see [3], pages [407–414], or [13]).  $\square$

*Remark 1.* We emphasize that the relations (3.5) and (3.6) depend only of the Cauchy data on  $\mathcal{I}^m$  for the evolution system  $(H_{\bar{g}}, H_\Phi, H_\rho)$  as shown in the next section.

**4. The Choice of Free Data and Construction of the Full Set of Initial Data for the Evolution System  $(H_{\bar{g}}, H_\Phi, H_\rho)$**  To proceed to the choice of the free data from which the full set of initial data of the evolution system  $(H_{\bar{g}}, H_\Phi, H_\rho)$  can be determined, it is necessary to give a complete description of the constraints (3.5)–(3.6) in terms of the Cauchy data of the evolution system  $(H_{\bar{g}}, H_\Phi, H_\rho)$ . Indeed, setting

$$(4.1) \quad \begin{aligned} \tilde{g}_{01}|_{\mathcal{I}^0} &= \theta, \quad \tilde{g}_{ab}|_{\mathcal{I}^0} = \Theta_{ab}, \quad \Phi|_{\mathcal{I}^0} = \phi, \quad \rho|_{\mathcal{I}^0} = \mathbf{f}, \quad \psi_{\mu\nu} = \frac{\partial \tilde{g}_{\mu\nu}}{\partial y^0}, \\ \partial_\mu &= \frac{\partial}{\partial y^\mu}, \quad d\tilde{p}' = d\tilde{p}^1 \dots d\tilde{p}^n, \end{aligned}$$

the Hamiltonian constraint  $\tilde{\mathfrak{C}}_{11} = 0$  and the momentum constraints  $\tilde{\mathfrak{C}}_{1a} = 0$ ,  $a = 2, \dots, n$ , reduce on  $\mathcal{I}^0$  respectively to the following partial differential relations of the Cauchy data  $(\theta, \Theta_{ab}, \psi_{11}, \mathbf{f}, \phi)$  respectively  $(\theta, \Theta_{ab}, \mathbf{f}, \phi, \psi_{1i})$ ,  $i = 1, \dots, n$ , of the evolution system  $(H_{\bar{g}}, H_{\Phi}, H_{\rho})$ , i.e.:

$$\begin{aligned} & \Theta^{cb} \partial_1 \Theta_{cb} \psi_{11} + 2\theta \partial_1 (\Theta^{ab} \partial_1 \Theta_{ab}) + \theta (\Theta^{cb} \partial_1 \Theta_{bd}) (\Theta^{de} \partial_1 \Theta_{ce}) - 2\partial_1 \theta \Theta^{ab} \partial_1 \Theta_{ab} \\ (4.2) \quad & = 4(\partial_1 \phi)^2 - 2 \int_{\mathbb{R}^n} \mathbf{f} \frac{|\theta| \sqrt{|\Theta|} (\mathbf{m}^2 + \Theta_{ab} \tilde{p}^a \tilde{p}^b)^2}{(\tilde{p}^1)^2} d\tilde{p}' ; \end{aligned}$$

$$\begin{aligned} & \partial_1 \psi_{1a} + \frac{\psi_{11} + \theta \Theta^{cd} \partial_1 \Theta_{cd} - \partial_1 \theta}{2\theta} \psi_{1a} + \frac{\psi_{11} \partial_a \theta}{2\theta} + (\Theta^{cb} \partial_1 \Theta_{ca}) \partial_b \theta \\ & - \frac{3}{2} \frac{(\partial_1 \theta)(\partial_a \theta)}{\theta} + \partial_{1a}^2 \theta - \theta \partial_c (\Theta^{cb} \partial_1 \Theta_{ab}) + \frac{1}{2} \partial_a (\Theta^{cd} \Theta_{cd}) - \theta^2 \partial_a \psi_{11} \\ & - \Theta^{cb} (\partial_c \theta) (\partial_1 \Theta_{ba}) - \frac{\theta}{2} \Theta^{de} \Theta^{cb} (\partial_1 \Theta_{ba}) (\partial_c \Theta_{de}) \\ & + \frac{\theta}{2} \Theta^{de} \Theta^{cb} (\partial_1 \Theta_{ab}) (\partial_a \Theta_{ec} + \partial_c \Theta_{ea} - \partial_e \Theta_{ac}) \\ (4.3) \quad & = 2\theta \left( \partial_1 \phi \partial_a \phi + \int_{\mathbb{R}^n} \frac{\mathbf{f} |\theta| \sqrt{|\Theta|} (\mathbf{m}^2 + \Theta_{cd} \tilde{p}^c \tilde{p}^d) \Theta_{ab} \tilde{p}^b}{(\tilde{p}^1)^2} d\tilde{p}' \right). \end{aligned}$$

The constraint  $\tilde{\mathfrak{C}}_{10} = 0$  is in turn a partial differential relation of the Cauchy data  $(\theta, \Theta_{ab}, \mathbf{f}, \phi, \psi_{1i}, \psi_{ab})$ ,  $i = 1, \dots, n$ ,  $a, b = 2, \dots, n$ , of the evolution system  $(H_{\bar{g}}, H_{\Phi}, H_{\rho})$  which, by setting  $\chi = \Theta^{ab} \psi_{ab}$ , reads

$$\begin{aligned} & \partial_1 \chi + \frac{1}{2} (\Theta^{cd} \partial_1 \Theta_{cd} - \frac{\psi_{11}}{\theta}) \chi - \Theta^{ab} \partial_a \psi_{1b} + \frac{1}{2} \Theta^{cd} \Theta^{ab} (2\partial_a \Theta_{cb} - \partial_c \Theta_{ab}) \psi_{1d} \\ & + \theta \Theta^{ab} \partial_{ab}^2 \ln |\theta| + \frac{\theta}{2} \Theta^{ab} \partial_b (\Theta^{cd} \partial_a \Theta_{cd}) - \frac{\theta}{2} \Theta^{ab} \partial_c [\Theta^{cd} (2\partial_a \Theta_{bd} - \partial_d \Theta_{ab})] \\ & + \frac{1}{2\theta} \Theta^{ab} (\psi_{1a} \psi_{1b} + \partial_a \theta \partial_b \theta) + \theta \Theta^{ab} \tilde{\Gamma}_{ac}^d \tilde{\Gamma}_{db}^c - \theta (\partial_c \ln |\theta| + \tilde{\Gamma}_{dc}^d) \Theta^{ab} \tilde{\Gamma}_{ab}^c \\ (4.4) \quad & = (\Theta^{cd} \partial_c \phi \partial_d \phi + V(\phi)) + 2 \int_{\mathbb{R}^n} \mathbf{f} \frac{\theta \sqrt{|\Theta|} (\mathbf{m}^2 + \Theta_{ab} \tilde{p}^a \tilde{p}^b)}{\tilde{p}^1} d\tilde{p}' ; \end{aligned}$$

with  $\tilde{\Gamma}_{ab}^c = \frac{1}{2} \Theta^{cd} (\partial_a \Theta_{bd} + \partial_b \Theta_{ad} - \partial_d \Theta_{ab})$ ,  $a, b, c, d = 2, \dots, n$ .

For the constraints's equations  $\tilde{\mathfrak{C}}_{ab} - \frac{\tilde{g}^{cd} \tilde{\mathfrak{C}}_{cd}}{n-1} \tilde{g}_{ab} = 0$ ,  $(a, b, c, d = 2, \dots, n)$ , extracted from (3.6), they are also partial differential relations of the Cauchy data  $(\theta, \Theta_{ab}, \mathbf{f}, \phi, \psi_{1i}, \psi_{ab})$ ,  $(i = 1, \dots, n, a, b = 2, \dots, n)$ , of the system  $(H_{\bar{g}}, H_{\Phi}, H_{\rho})$ ,

i.e.:

$$\begin{aligned}
& \partial_1 \psi_{ab} + \frac{1}{2} \left( -\frac{\psi_{11}}{\theta} + \Theta^{ef} \partial_1 \Theta_{ef} \delta_a^c \delta_b^d - \frac{1}{\theta} (\Theta^{ce} \partial_1 \Theta_{ea} \delta_b^d + \Theta^{ed} \partial_1 \Theta_{eb} \delta_a^c) \right) \psi_{cd} \\
& - \frac{\theta}{2} \partial_c (\Theta^{cd} (\partial_b \Theta_{da} + \partial_a \Theta_{db} - \partial_d \Theta_{ab})) \\
& + \frac{1}{2} \Theta^{cd} \psi_{1d} (\partial_b \Theta_{ca} + \partial_a \Theta_{cb} - \partial_c \Theta_{ba}) \\
& - \frac{1}{2} (\partial_b \psi_{1a} - \partial_a \psi_{1b}) + \theta \partial_{ab}^2 \ln |\theta| + \frac{\theta}{2} \partial_b (\Theta^{cd} \partial_a \Theta_{cd}) \\
& + \frac{1}{4\theta} \psi_{1a} (2\psi_{1b} - \partial_b \theta) \\
& + \tilde{\Gamma}_{ac}^d \tilde{\Gamma}_{db}^c + \frac{\theta}{2} \partial_1 \Theta_{ab} \chi - \theta (\partial_c \ln |\theta| + \tilde{\Gamma}_{dc}^d) \tilde{\Gamma}_{ab}^c \\
(4.5) \quad & = \frac{\tilde{R}^{(n-1)} - \Theta^{cd} \tilde{T}_{cd}}{n-1} \Theta_{ab} + \tilde{T}_{ab}; \quad a, b, c, d, e, f = 2, \dots, n;
\end{aligned}$$

$$\begin{aligned}
\tilde{R}^{(n-1)} \quad & \equiv \quad \Theta^{cd} \tilde{R}_{cd} = \frac{\partial_1 \chi}{\theta} + \frac{1}{2\theta} (\Theta^{cd} \partial_1 \Theta_{cd} - \frac{\psi_{11}}{\theta}) \chi \\
& - \frac{\Theta^{ab} \partial_a \psi_{1b}}{\theta} + \Theta^{ab} \partial_{ab}^2 \ln |\theta| \\
& + \frac{1}{2\theta} \Theta^{cd} \Theta^{ab} (2\partial_a \Theta_{cb} - \partial_c \Theta_{ab}) \psi_{1d} + \frac{1}{2} \Theta^{ab} \partial_b (\Theta^{cd} \partial_a \Theta_{cd}) \\
& - \frac{1}{2} \Theta^{ab} \Theta^{cd} (2\partial_a \Theta_{bd} - \partial_d \Theta_{ab}) \\
(4.6) \quad & + \frac{1}{2\theta^2} \Theta^{ab} (\psi_{1a} \psi_{1b} + \partial_a \theta \partial_b \theta) + \Theta^{ab} \tilde{\Gamma}_{ac}^d \tilde{\Gamma}_{db}^c - (\partial_c \ln |\theta| + \tilde{\Gamma}_{dc}^d) \Theta^{ab} \tilde{\Gamma}_{ab}^c;
\end{aligned}$$

$$\begin{aligned}
\tilde{T}_{ab} \quad & = \quad \partial_a \phi \partial_b \phi - \frac{1}{2} \Theta_{ab} \left( \frac{2}{\theta} \left[ \frac{\partial \Phi}{\partial y^0} \right] (\partial_1 \phi) + \Theta^{cd} \partial_c \phi \partial_d \phi + V(\phi) \right) \\
(4.7) \quad & - \int_{\mathbb{R}^n} \mathbf{f} \frac{2\theta \sqrt{|\Theta|} \Theta_{ae} \Theta_{bf} \tilde{p}^e \tilde{p}^f}{\tilde{p}^1} d\tilde{p}' .
\end{aligned}$$

The last constraint  $\tilde{\mathcal{C}}_{00} - 2\frac{\tilde{g}_{01}}{n-1} \tilde{\mathcal{C}} + \tilde{g}_{01} \frac{\partial(\tilde{\Gamma}^0 + \tilde{\Gamma}^1)}{\partial y^0} = 0$  is a partial differential relation

comprising all the Cauchy data of the evolution system  $(H_{\bar{g}}, H_{\Phi}, H_{\rho})$ , i.e.:

$$\begin{aligned}
\partial_1 \psi_{01} &- \frac{1}{2} \left( \frac{\chi}{2} + \partial_1 \ln |\theta| \right) \psi_{01} + \frac{\theta}{2} \partial_1 \chi - \frac{3\theta}{2} \Theta^{cd} \psi_{1d} \partial_c \theta + \frac{\theta}{2} (\partial_d \Theta^{dc}) \partial_c \theta \\
&+ \frac{\theta}{2} \Theta^{cd} \partial_{cd}^2 \theta - \frac{3}{4} \Theta^{cb} \psi_{1b} \psi_{1c} + \frac{1}{4} \Theta^{cb} \partial_b \theta \partial_c \theta - \frac{\chi}{4} \psi_{11} + \frac{\theta}{2} \Theta^{cb} \tilde{\Gamma}_{cd}^d \partial_b \theta \\
&- \frac{1}{2} (\partial_1 \ln |\theta|) \psi_{11} - \frac{1}{8} (\Theta^{cb} \partial_1 \Theta_{cb}) \psi_{11} + \frac{\partial_1 \psi_{11}}{2} - \frac{\psi_{11}^2}{2\theta} - \frac{\theta}{4} \Theta^{ab} \partial_1 \psi_{ab} \\
&- \frac{\psi_{11} \partial_1 \theta}{2\theta} - \Theta^{ad} \psi_{1d} \psi_{1a} + \frac{\theta}{2} \Theta^{ac} \Theta^{bd} (\partial_1 \Theta_{ab}) \psi_{cd} \\
(4.8) \quad &= \frac{\theta}{2} \left( \left[ \frac{\partial \Phi}{\partial y^0} \right]^2 - 2 \int_{\mathbb{R}^n} \mathbf{f} |\theta|^3 \sqrt{|\Theta|} \tilde{p}^1 d\tilde{p}' \right) - \frac{\theta^2}{2} (\tilde{R}^{(n-1)}) + 2 \frac{\Theta^{ab} \tilde{T}_{ab}}{n-1}.
\end{aligned}$$

We note that a similar description of the constraints (3.5)–(3.6) (for  $m = 1$ ) in terms of the Cauchy data  $(\underline{\theta}, \underline{\Theta}_{ab}, \underline{\psi}_{1\nu}, \underline{\psi}_{ab}, \underline{\phi}, \underline{\mathbf{f}})$  on  $\mathcal{I}^1 \times \mathbb{R}^n$  of the evolution system  $(H_{\bar{g}}, H_{\Phi}, H_{\rho})$  is also valid, where:

$$(4.9) \quad \underline{\theta} := \tilde{g}_{01}|_{\mathcal{I}^1}, \quad \underline{\Theta}_{ab} := \tilde{g}_{ab}|_{\mathcal{I}^1}, \quad \underline{\phi} := \Phi|_{\mathcal{I}^1}, \quad \underline{\mathbf{f}} := \rho|_{\mathcal{I}^1}, \quad \underline{\psi}_{\mu\nu} := \frac{\partial \tilde{g}_{\mu\nu}}{\partial y^1}.$$

**The free data** The free data making possible the resolution of the constraints (3.5)–(3.6) comprise:

(a)  $\mathcal{C}^\infty$  functions  $\tilde{\gamma}_{ab} = \gamma_{ab}(y^0 - y^1, y^a)$  where the  $\gamma_{ab} \equiv \gamma_{ab}(x^1, x^a)$  are  $\mathcal{C}^\infty$  functions of the variables  $x^1, x^a$  that make up a symmetric positive definite matrix satisfying  $\left| \gamma^{ab} \frac{\partial \gamma_{ab}}{\partial x^1} \right| > 0$ ;

(b) smooth functions  $(\theta, \phi)$  on  $\mathcal{I}^0$ , and  $\mathbf{f}$  on  $\mathcal{I}^0 \times \mathbb{R}^n$  (respectively  $(\underline{\theta}, \underline{\phi})$  on  $\mathcal{I}^1$ , and  $\underline{\mathbf{f}}$  on  $\mathcal{I}^1 \times \mathbb{R}^n$ ) and such that  $\theta, \underline{\theta}$  are negative,  $\mathbf{f}$  (respectively  $\underline{\mathbf{f}}$ ) is non negative of compact support contained in  $\{\tilde{p}^1 > c_1 > 0\}$  (respectively  $\{\tilde{p}^0 > c_0 > 0\}$ ) for a mass  $\mathbf{m} \neq 0$ ; and for the zero mass the support of  $\mathbf{f}$  (respectively  $\underline{\mathbf{f}}$ ) is contained in  $\{\tilde{p}^1 > c_1 > 0, \sum_{a=2}^n (\tilde{p}^a)^2 > c'_2 > 0\}$  (respectively  $\{\tilde{p}^0 > c_0 > 0, \sum_{a=2}^n (\tilde{p}^a)^2 > c_2 > 0\}$ ), besides that,  $Supp(\mathbf{f}) \cap (\mathcal{S} \times \mathbb{R}^n) = \emptyset$ ,  $Supp(\underline{\mathbf{f}}) \cap (\mathcal{S} \times \mathbb{R}^n) = \emptyset$ ; and one has compatibilities relations

$$(4.10) \quad \theta = \underline{\theta}, \quad \phi = \underline{\phi}, \quad \text{on } \mathcal{S}, \quad \mathbf{f} = \underline{\mathbf{f}} \quad \text{on } \mathcal{S} \times \mathbb{R}^n.$$

**THEOREM 2.** *Given the free data as described above by (a)–(b), then, there exists a unique global solution  $(\theta, \Theta_{ab}, \psi_{1\nu}, \psi_{ab}, \phi, \mathbf{f})$  on  $\mathcal{I}^0 \times \mathbb{R}^n$  and  $(\underline{\theta}, \underline{\Theta}_{ab}, \underline{\psi}_{1\nu}, \underline{\psi}_{ab}, \underline{\phi}, \underline{\mathbf{f}})$  on  $\mathcal{I}^1 \times \mathbb{R}^n$  of the initial data constraints (3.5)–(3.6) for the Einstein-Vlasov scalar field equations.*

**PROOF.** We concentrate on the case of  $\mathcal{I}^0 \times \mathbb{R}^n$  and an analogous scheme holds on  $\mathcal{I}^1 \times \mathbb{R}^n$ .

Given the free data (a)–(b), one solves the constraints. Indeed, for the case of  $\mathcal{I}^0$ , let set  $\Theta_{ab}(y^1, y^a) = \gamma_{ab}(-y^1, y^a)$ ,  $\underline{\Theta}_{ab}(y^0, y^a) = \gamma_{ab}(y^0, y^a)$ ; then

$|\Theta^{ab}\partial_1\Theta_{ab}| > 0$ , and  $\psi_{11}$  solves algebraically the Hamiltonian constraint  $\tilde{\mathfrak{C}}_{11} = 0$  as described by (4.2). The other constraints (4.3)–(4.8) are hierarchical linear ordinary differential equations of the variable  $y^1$  depending smoothly on the parameters  $y^a$ ,  $a = 2, \dots, n$ . They are solved hierarchically via the theory of ordinary differential systems, using the initial conditions

$$(4.11) \quad \begin{aligned} \psi_{ab|S} &= \frac{\partial\gamma_{ab}}{\partial x^1|_S}, \quad \underline{\psi}_{ab|S} = -\frac{\partial\gamma_{ab}}{\partial x^1|_S}, \quad \psi_{01|S} = \frac{\partial\theta}{\partial y^0|_S}, \quad \underline{\psi}_{01|S} = \frac{\partial\theta}{\partial y^1|_S}, \\ \psi_{1i|S} &= 0, \quad \underline{\psi}_{0i|S} = 0; \end{aligned}$$

one obtains a unique global solution  $(\psi_{1a}, \psi_{ab}, \psi_{01}), a, b = 2, \dots, n$ . Concretely, one considers the momentum constraint  $\tilde{\mathfrak{C}}_{01} = 0$  described in (4.4) and the constraints  $Z_{ab} \equiv \tilde{\mathfrak{C}}_{ab} - \frac{\tilde{g}^{cd}\tilde{\mathfrak{C}}_{cd}}{n-1}\tilde{g}_{ab} = 0$  described in (4.5) for  $(a, b) \neq (2, 2)$  since  $(Z_{ab}), (a, b = 2, \dots, n)$ , is a traceless tensor. One can first solve the constraint (4.4) of unknown  $\chi = \Theta^{ab}\psi_{ab}$ . After that one solves the constraints (4.5) of unknowns  $\psi_{ab}$  for  $(a, b) \neq (2, 2)$  provided  $\psi_{22}$  takes the value  $\psi_{22} = \frac{1}{\Theta^{22}}(\chi - \sum_{(a,b) \neq (2,2)} \Theta^{ab}\psi_{ab})$ , and where  $\Theta^{ab}\tilde{R}_{ab} \equiv \tilde{R}^{(n-1)}$  equals  $-2\frac{\tilde{T}_{01}}{\theta}$  since  $\tilde{\mathfrak{C}}_{01} = 0$  is

satisfied. That  $Z_{22} = 0$  is also satisfied with  $\psi_{22} = \frac{1}{\Theta^{22}}(\chi - \sum_{(a,b) \neq (2,2)} \Theta^{ab}\psi_{ab})$

follows from the traceless property of  $(Z_{ab})$ . For the outgoing derivative  $\left[\frac{\partial\Phi}{\partial y^0}\right]$  which appears in the constraints (4.5), (4.8), it is obtained by solving, using the initial datum  $\left[\frac{\partial\Phi}{\partial y^0}\right]_S = \frac{\partial\phi}{\partial y^0}(0, y^a)$ , the propagation equation obtained by restricting the equation  $H_\Phi$  to  $\mathcal{I}^0$ . At least the constraint (4.8) determines  $\psi_{01}$ .  $\square$

### 5. Resolution of the Evolution System $(H_{\bar{g}}, H_\Phi, H_\rho)$

**THEOREM 3.** *Given the free data  $\gamma = (\gamma_{ab})$  on  $\mathbb{R}^n$ ,  $(\theta, \phi)$  on  $\mathcal{I}^0$ , and  $\mathbf{f}$  on  $\mathcal{I}^0 \times \mathbb{R}^n$  (respectively  $(\underline{\theta}, \underline{\phi})$  on  $\mathcal{I}^1$ , and  $\underline{\mathbf{f}}$  on  $\mathcal{I}^1 \times \mathbb{R}^n$ ); as described in Section 4. Then, there exists a unique 4-tuple  $(\mathcal{V}, g, \Phi, \rho)$  such that:  $\mathcal{V}$  is a neighborhood of  $\mathcal{S}$  in  $\mathcal{Y} := \{(y^\mu) \in \mathbb{R}^{n+1} / y^0 \geq 0, y^1 \geq 0\}$ ;  $g$  is a Lorentzian metric on  $\mathcal{V}$  of the form*

$$(5.1) \quad g = -\frac{|\bar{g}|}{|\gamma|}(dx^0)^2 + g_{ij}dx^i dx^j; \quad \bar{g} = (g_{ij});$$

*$(g, \Phi, \rho)$  is a  $C^\infty$  solution of the Einstein-Vlasov scalar field equations in  $\mathcal{V} \times \{(p^\mu) \in \mathbb{R}^{n+1} / g_{\mu\nu}p^\mu p^\nu = -m^2, p^0 > 0\}$  with  $\rho$  of compact support; and  $\tilde{g}_{01}|_{\mathcal{I}^0} = \theta$ ,  $\tilde{g}_{01}|_{\mathcal{I}^1} = \underline{\theta}$ ,  $\tilde{g}_{ab}|_{\mathcal{I}^0} = \gamma_{ab}(-y^1, y^a)$ ,  $\tilde{g}_{ab}|_{\mathcal{I}^1} = \gamma_{ab}(y^0, y^a)$ ,  $\Phi|_{\mathcal{I}^0} = \phi$ ,  $\Phi|_{\mathcal{I}^1} = \underline{\phi}$ ,  $\rho|_{(\mathcal{I}^0 \times \mathbb{P}_x)} = \mathbf{f}$ ,  $\rho|_{(\mathcal{I}^1 \times \mathbb{P}_x)} = \underline{\mathbf{f}}$ .*

**SKETCH OF THE PROOF.** (See Appendix A for a more extended version of the proof.) Let us denote by  $(\bar{g}_0, k_0, \rho_0, \phi_0)$  the solution of the initial data constraints

problem as constructed in Section 4, from the free data. To solve for the initial data  $(\bar{g}_0, k_0, \rho_0, \phi_0)$ , the evolution system  $(H_{\bar{g}}, H_{\Phi}, H_{\rho})$  in the domain  $\widehat{Y} := Y \times \mathbb{R}^n$ , we first proceed to the unique determination of the restrictions to the initial hypersurfaces  $\mathcal{I}^0$  and  $\mathcal{I}^1$  of the derivatives of all order of the possible  $\mathcal{C}^\infty$  solution  $(\bar{g}, \Phi, \rho)$ . Then, by using some variants of Borel's classical lemma and some arguments of domain of dependence, we can transform the evolution problem into a spacelike Cauchy problem  $(\mathcal{P})$  for a third order hyperbolic system of unknown  $(h, f, A)$  defined in the domain  $\widehat{\Omega}_T \equiv \Omega_T \times \mathbb{R}^n$ , with zero initial data on the spacelike hypersurface  $\Lambda_0$ , where for  $T > 0$ ,  $0 \leq t \leq T$ ,

$$(5.2) \quad \begin{aligned} \Omega_t &:= \{(x^\alpha) \in \mathbb{R}^{n+1} / 0 < x^0 < t; |x^i| < K(2t - x^0); i = 1, 2, \dots, n\}, \\ \Lambda_\tau &:= \Omega_t \cap \{x^0 = \tau\}, \quad 0 \leq \tau \leq t, \end{aligned}$$

with  $K > 1$  large enough and suppose that the hypersurfaces

$$(5.3) \quad \mathcal{H}^i := \{(x^\alpha) \in \mathbb{R}^{n+1} / |x^i| = K(2t - x^0), \quad 0 \leq x^0 \leq t, \quad i = 1, 2, \dots, n\}$$

are spacelike with respect to the constructed metric

$$(5.4) \quad g_0 = -\frac{|\bar{g}_0|}{|\gamma|} (dx^0)^2 + \bar{g}_{0ij} dx^i dx^j.$$

The linearized  $\mathcal{C}^\infty$  problem  $(\mathcal{P}_l)$  associated to  $(\mathcal{P})$  is solved by applying the Leray's theory of hyperbolic systems and the classical method of characteristics.

Let  $s$  the smallest integer such that  $s > \frac{n}{2} + 2$ , there exists a suitable weighted Sobolev space  $\mathcal{E}^s(\widehat{\Omega}_T)$  of order  $s$ , in which one can develop a fixed point method based on some energy estimates established for the  $\mathcal{C}^\infty$  solution of the linearized problem  $(\mathcal{P}_l)$ , and which leads to a unique solution  $(h, f, A) \in \mathcal{E}^s(\widehat{\Omega}_T)$  of the problem  $(\mathcal{P})$ , for  $T > 0$  small enough. Then the  $\mathcal{C}^\infty$  regularity of this solution  $(h, f, A)$  is established by showing by induction on  $m$  and some classical arguments [4], that  $(h, f, A) \in \mathcal{E}^m(\widehat{\Omega}_T)$  for every  $m \geq s$ , and finally by using the Sobolev embedding theorem. The evolution system  $(H_{\bar{g}}, H_{\rho}, H_{\Phi})$  thus has a unique  $\mathcal{C}^\infty$  solution  $(\bar{g}, \rho, \Phi)$  for the deduced initial data  $(\bar{g}_0, k_0, \rho_0, \phi_0)$ .  $\square$

**Appendix A Sketch of the Proof of Theorem 3** Given the full initial data  $(\bar{g}_0, k_0, \rho_0, \phi_0)$  as constructed, in Section 4, as the solution of the initial data constraints's problem, we consider now the characteristic Cauchy problem

$$(A.1) \quad \mathcal{P} \begin{cases} H_{\bar{g}} : \partial_0 R_{ij} - \bar{\nabla}_i R_{j0} - \bar{\nabla}_j R_{i0} = \partial_0 \Lambda_{ij} - \bar{\nabla}_i \Lambda_{j0} - \bar{\nabla}_j \Lambda_{i0}, & \text{in } Y, \\ H_{\rho} : p^\alpha \frac{\partial \rho}{\partial x^\alpha} - \Gamma_{\mu\nu}^i p^\mu p^\nu \frac{\partial \rho}{\partial p^i} = 0, & \text{in } \mathbb{P}, \\ H_{\Phi} : \square_g \Phi = V'(\Phi), & \text{in } Y, \\ (\bar{g}, \partial_0 \bar{g}, \rho, \Phi)|_{\mathcal{I}} = (\bar{g}_0, k_0, \rho_0, \phi_0). \end{cases}$$

A first step towards the solving of the problem  $\mathcal{P}$  consists to determine uniquely, by induction on  $k \in \mathbb{N}$ , the functions  $\psi^{(k)}$  defined as the trace on the initial hypersurface  $\mathcal{I}$  of the derivatives  $\left(\frac{\partial^k \bar{g}}{(\partial x^0)^k}, \frac{\partial^k \rho}{(\partial x^0)^k}, \frac{\partial^k \Phi}{(\partial x^0)^k}\right)$  of the possible  $C^\infty$  solution  $(\bar{g}, \rho, \Phi)$  of the problem  $\mathcal{P}$ . Using subsequently some variants of Borel's classical lemma, we can construct an auxiliary function  $w = (\mu = (\mu_{ij}), \varrho, \kappa) \in C^\infty(\Omega_T \times \mathbb{R}^n)$  s.t.  $\left(\frac{\partial^k \mu}{(\partial x^0)^k}, \frac{\partial^k \varrho}{(\partial x^0)^k}, \frac{\partial^k \kappa}{(\partial x^0)^k}\right)_{/\mathcal{I}} = \psi^{(k)}$  for every  $k \in \mathbb{N}$ , where, for  $0 \leq t \leq T$ ,  $0 \leq \tau \leq t$ ,  $\Omega_t$ ,  $\Lambda_\tau$  are defined in (5.2). The function  $w$  verifies thus on  $\mathcal{I}$  the evolution system  $(H_{\bar{g}}, H_\rho, H_\Phi)$  and its derivatives of all orders. Introducing now the new unknown  $v = u - w = (\tilde{h}, \tilde{A}, \tilde{f})$  with  $\tilde{h} = \bar{g} - \mu$ ,  $\tilde{f} = \rho - \varrho$ ,  $\tilde{A} = \Phi - \kappa$ , we can transform the problem  $\mathcal{P}$  into a zero initial data characteristic Cauchy problem  $\mathcal{P}_1$  in  $\mathbb{P}$ , which is of the form

$$(A.2) \quad \mathcal{P}_1 \begin{cases} H_{\tilde{h}} : \tilde{g}^{\lambda\nu} D_{\lambda\nu} \partial_0 \tilde{h}_{ij} = \tilde{F}_{ij}(x, D^\alpha \tilde{h}_{lk}, D^\beta \tilde{f}, D^\alpha \tilde{A}), |\alpha| \leq 2, \\ H_{\tilde{f}} : \tilde{p}^\alpha \frac{\partial \tilde{f}}{\partial x^\alpha} + \tilde{P}^i \frac{\partial \tilde{f}}{\partial p^i} + \mathcal{L}_Z \varrho = 0, Z = (\tilde{p}^\nu, \tilde{P}^i) \\ H_{\tilde{A}} : \tilde{g}^{\lambda\nu} D_{\lambda\nu} \tilde{A} = \tilde{F}(x, D^\beta \tilde{h}_{lk}, D^\gamma \tilde{A}), |\beta| \leq 1, |\gamma| \leq 1 \\ (\partial_0^k \tilde{h}, \partial_0^k \tilde{f}, \partial_0^k \tilde{A})_{/\mathcal{I}} = (0), \forall k \in \mathbb{N}; \end{cases}$$

where

$$\begin{aligned} \alpha &= (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{N}^{n+1}, \quad |\alpha| = \alpha_0 + \alpha_1 + \dots + \alpha_n, \\ D^\alpha &\equiv \frac{\partial^{|\alpha|}}{(\partial x^0)^{\alpha_0} (\partial x^1)^{\alpha_1} \dots (\partial x^n)^{\alpha_n}}, \end{aligned}$$

$\mathcal{L}_Z$  denotes the Lie-derivative w.r.t.  $Z \equiv (\tilde{p}^\nu, \tilde{P}^l)$  with

$$\begin{aligned} \tilde{p}^0 &= \sqrt{|\gamma|} \frac{\sqrt{\mathbf{m}^2 + \tilde{g}_{ij} p^i p^j}}{\sqrt{|\tilde{g}|}}, \tilde{p}^i = p^i, \tilde{P}^l = -\tilde{\Gamma}^l_{\lambda\nu} p^\lambda p^\nu, \\ \tilde{g}_{ij} &= \tilde{h}_{ij} + \mu_{ij}, \tilde{g} = (\tilde{g}_{ij}), \tilde{g}_{00} = -\frac{|\tilde{g}|}{|\gamma|}, \gamma = (\gamma_{ab}), a, b = 2, \dots, n. \end{aligned}$$

The characteristic Cauchy problem  $\mathcal{P}_1$  is then extended to an ordinary spacelike Cauchy problem  $(\mathcal{P})$  in  $\Omega_T \times \mathbb{R}^n$  with zero initial data on the the spacelike hypersurface  $\Lambda_0$ , i.e.:

$$(A.3) \quad (\mathcal{P}) \begin{cases} H_h : \tilde{g}^{\lambda\nu} D_{\lambda\nu} \partial_0 h_{ij} = \tilde{F}_{ij} + r_{ij} & \text{in } \Omega_T \\ H_f : \tilde{p}^\alpha \frac{\partial f}{\partial x^\alpha} + \tilde{P}^i \frac{\partial f}{\partial p^i} + \mathcal{L}_Z \varrho = r & \text{in } \Omega_T \times \mathbb{R}^n \\ H_A : \tilde{g}^{\lambda\nu} D_{\lambda\nu} A = \tilde{F} + r_\Phi & \text{in } \Omega_T \\ (\partial_0^q h, f, A, \partial_0 A)_{/\Lambda_0} = 0, q \in \{0, 1, 2\}. \end{cases}$$

The unknowns functions are  $h_{ij}, A, f$ ; the  $C^\infty$  functions  $r_{ij}, r_\Phi, r$  are defined by

$$(A.4) \quad \begin{aligned} r_{ij} &= \begin{cases} 0, & \text{if } (x^\alpha) \in Y \\ -\tilde{F}_{ij}(x^\alpha, 0, \dots, 0), & \text{if } (x^\alpha) \in \Omega_T - Y, \end{cases} \\ r_\Phi &= \begin{cases} 0, & \text{if } (x^\alpha) \in Y \\ -\tilde{F}(x^\alpha, 0, \dots, 0), & \text{if } (x^\alpha) \in \Omega_T - Y, \end{cases} \end{aligned}$$

$$(A.5) \quad r = \begin{cases} 0, & \text{if } (x^\alpha, p^\delta) \in \mathbb{P} \\ \mathcal{L}_{Z_0} \varrho(x^\alpha, 0, \dots, 0), & \text{if } (x^\alpha) \in (\Omega_T \times \mathbb{R}^n) - \mathbb{P}, \end{cases}$$

where  $Z_0$  is deduced of  $Z$  by imposing  $\tilde{h} = 0$ .

Now, for given  $C^\infty$  functions  $\hat{h} = (\hat{h}_{ij})$ ,  $\hat{A}$  on  $\Omega_T$ ,  $\hat{f}$  on  $\Omega_T \times \mathbb{R}^n$  such that  $\bar{v} = (v_{ij} = \mu_{ij} + \hat{h}_{ij})$  is properly riemannian on  $\Omega_T$ , and for

$$v = -\frac{|\bar{v}|}{|\gamma|} (dx^0)^2 + v_{ij} dx^i dx^j$$

we consider the linear problem

$$(A.6) \quad \begin{cases} H_h^l : v^{\lambda\nu} (x^\alpha) D_{\lambda\nu} \partial_0 h_{ij} = \tilde{F}_{ij} + r_{ij} \equiv G_{ij} & \text{in } \Omega_T \\ H_f^l : \sqrt{|\gamma|} \frac{\sqrt{\mathbf{m}^2 + v_{ij} p^i p^j}}{\sqrt{|\bar{v}|}} \frac{\partial f}{\partial x^0} + p^i \frac{\partial f}{\partial x^i} + \tilde{P}^i \frac{\partial f}{\partial p^i} = -\mathcal{L}_{Z_0} \varrho + r & \text{in } \Omega_T \times \mathbb{R}^n \\ H_A^l : v^{\lambda\nu} (x^\alpha) D_{\lambda\nu} A = \tilde{F} + r_\Phi \equiv J & \text{in } \Omega_T \\ (\partial_0^q h, f, A, \partial_0 A)_{/\Lambda_0} = 0, q \in \{0, 1, 2\}. \end{cases}$$

where the  $C^\infty$  functions  $\tilde{P}^i$ ,  $G \equiv (G_{ij} = \tilde{F}_{ij} + r_{ij})$ ,  $J \equiv \tilde{F} + r_\Phi$ ,  $-\mathcal{L}_{Z_0} \varrho + r$  are taken for the metric  $v = (v_{00}, \bar{v} = (v_{ij}))$ , the function  $\hat{A}$ , and the function  $\hat{f}$  which is supposed of compact support. Thanks to the Leray theory of hyperbolic systems applied for the equations  $H_h^l, H_A^l$ , and the classical method of characteristics applied for the equation  $H_f^l$ , one has:

LEMMA 1. *The hypotheses are those of the theorem, then, for given  $C^\infty$  functions  $\hat{h} = (\hat{h}_{ij})$ ,  $\hat{A}$  on  $\Omega_T$ ,  $\hat{f}$  on  $\Omega_T \times \mathbb{R}^n$ , the linear problem (P<sub>l</sub>) has a unique  $C^\infty$  solution defined on  $\Omega_T \times \mathbb{R}^n$ , with support contained in  $Y \times \mathbb{R}^n$ .*

Now, to obtain a  $C^\infty$  solution for the nonlinear problem (P), we consider a framework of weighted Sobolev spaces in order to apply a fixed point method. Given  $T > 0$  and  $K > 1$  large enough as indicated above, considering  $\Omega_t, \Lambda_t$  as defined in (5.2), one sets for  $0 \leq t \leq T$ :

$$(A.7) \quad P_t := \{(x^\alpha, p^\delta) \in \Omega_t \times \mathbb{R}^{n+1} / v_{\mu\nu} p^\mu p^\nu = -\mathbf{m}^2\}.$$

One denotes by  $C_0^\infty(\Omega_t)$  the space of restrictions to  $\Omega_t$  of  $C^\infty$  functions defined in a neighborhood of  $\Omega_t$  and by  $C_0^\infty(P_t)$  the space of restrictions to  $P_t$  of  $C^\infty$  functions with compact support in a neighborhood of  $P_t$ .

Let  $s \in \mathbb{N}$ . For a function  $v = (v_I)$  defined in a neighborhood of  $\Omega_T$ , for a function  $f$  defined in a neighborhood of  $P_T$ , we introduce the norms

$$\begin{aligned} \|v\|_{H^s(\Lambda_t)}^2 &:= \sum_I \sum_{|\alpha| \leq s} \int_{\Lambda_t} |D^\alpha v_I|^2 dx, \quad \|f\|_{F^s(\Lambda_t \times \mathbb{R}^n)}^2 \\ &:= \sum_{|\alpha|+|\beta| \leq s} \int_{\Lambda_t \times \mathbb{R}^n} (p^0)^{2(|\alpha|+|\beta|)+1} |D_x^\alpha \partial_p^\beta f|^2 dx' dp; \end{aligned}$$

where for  $\alpha \in \mathbb{N}^{n+1}$ ,  $\beta \in \mathbb{N}^n$ ,

$$\begin{aligned} D_x^\alpha \partial_p^\beta &:= \frac{\partial^{|\alpha|+|\beta|}}{(\partial x^0)^{\alpha_0} (\partial x^1)^{\alpha_1} \dots (\partial x^n)^{\alpha_n} (\partial p^1)^{\beta_1} (\partial p^2)^{\beta_2} \dots (\partial p^n)^{\beta_n}}, \\ dx' &:= dx^1 \dots dx^n, \quad dp := dp^1 \dots dp^n, \quad dx = dx^0 dx'; \end{aligned}$$

and denote by  $E^s(\Omega_t)$  (respectively  $E^s(P_t)$ ) the closure of  $C_0^\infty(\Omega_t)$  (respectively  $C_0^\infty(P_t)$ ) w.r.t. the norms

$$\|v\|_{E^s(\Omega_t)} := \sup_{0 \leq \tau \leq t} \|v\|_{H^s(\Lambda_\tau)}, \quad \|f\|_{E^s(P_t)} := \sup_{0 \leq \tau \leq t} \|f\|_{F^s(\Lambda_\tau \times \mathbb{R}^n)}.$$

LEMMA 2. *For every solution  $(h, f, A) \in C_0^\infty(\Omega_T) \times C_0^\infty(\Omega_T \times \mathbb{R}^n) \times C_0^\infty(\Omega_T)$  of the linearized problem  $(\mathcal{P}_l)$ , one has for  $s > \frac{n}{2} + 2$ ,  $t \in ]0, T[$ :*

$$\begin{aligned} \|f\|_{E^s(P_t)} &\leq C(T) \| -\mathcal{L}_Z \varrho + r \|_{E^s(P_t)} \cdot t; \\ \|h\|_{H^s(\Lambda_t)}^2 &\leq \int_0^t \left\{ R_1(\tau) \|h\|_{H^s(\Lambda_\tau)} + R_2(\tau) \|h\|_{H^s(\Lambda_\tau)}^2 \right\} d\tau; \\ \|h\|_{E^s(\Omega_t)} &\leq Z_2 \|G\|_{E^{s-2}(\Omega_t)} \cdot t; \\ \|A\|_{H^s(\Lambda_t)}^2 &\leq \int_0^t \left\{ R_3(\tau) \|h\|_{H^s(\Lambda_\tau)} + R_4(\tau) \|h\|_{H^s(\Lambda_\tau)}^2 \right\} d\tau; \\ \|A\|_{E^s(\Omega_t)} &\leq Z_3 \|J\|_{E^{s-1}(\Omega_t)} \cdot t; \end{aligned}$$

$Z_2, Z_3$  are constants depending only of  $T$  and some intrinsic constants.

Now we consider the map

$$\begin{aligned} \mathbf{L} &: (C_0^\infty(\Omega_T))^{\frac{n(n+1)}{2}} \times C_0^\infty(\Omega_T \times \mathbb{R}^n) \times C_0^\infty(\Omega_T) \\ &\rightarrow (C_0^\infty(\Omega_T))^{\frac{n(n+1)}{2}} \times C_0^\infty(\Omega_T \times \mathbb{R}^n) \times C_0^\infty(\Omega_T) \\ &\quad (\widehat{h} = (\widehat{h}_{ij}), \widehat{f}, \widehat{A}) \mapsto (h, f, A) \end{aligned}$$

where  $(h, f, A)$  is the unique solution of the linearized problem  $(\mathcal{P}_l)$  associated to  $(\mathcal{P})$  for the given functions  $(\widehat{h} = (\widehat{h}_{ij}), \widehat{f}, \widehat{A})$ ; let  $s$  denote the smallest integer such that  $s > \frac{n}{2} + 2$ ; set:  $\mathcal{E}^s \equiv (E^s(\Omega_T))^{\frac{n(n+1)}{2}} \times E^s(\Omega_T \times \mathbb{R}^n) \times E^s(\Omega_T)$ . Therefore  $\mathbf{L}$  extends to a map  $\mathbf{L}'$  defined similarly and mapping  $\mathcal{E}^s$  into itself:

$$\begin{aligned} \mathbf{L}' : \mathcal{E}^s &\equiv (E^s(\Omega_T))^{\frac{n(n+1)}{2}} \times E^s(\Omega_T \times \mathbb{R}^n) \times E^s(\Omega_T) \\ &\rightarrow \mathcal{E}^s \equiv (E^s(\Omega_T))^{\frac{n(n+1)}{2}} \times E^s(\Omega_T \times \mathbb{R}^n) \times E^s(\Omega_T) \\ &\quad (\widehat{h} = (\widehat{h}_{ij}), \widehat{f}, \widehat{A}) \mapsto (h, f, A). \end{aligned}$$

Using the energy estimates above, one shows that there exist  $T_* > 0$  small enough and  $R > 0$  large enough and suppose that  $\mathbf{L}'$  is a contraction map from the closed ball  $B(0, R)$  of the Banach space  $\mathcal{E}^s$  into itself.  $\mathbf{L}'$  admits a fixed point which is the desired solution of  $(\mathcal{P})$ . The  $\mathcal{C}^\infty$  regularity of this solution  $(h, f, A)$  is established by showing by induction on  $m$  and some classical arguments [4], that  $(h, f, A) \in \mathcal{E}^m(\Omega_T \times \mathbb{R}^n)$  for every  $m \geq s$ , and finally by using the Sobolev embedding theorem. The support of this solution is in the domain above  $\mathcal{I}$ . The evolution system  $(H_{\bar{g}}, H_\Phi, H_\rho)$  thus has a unique  $\mathcal{C}^\infty$  solution  $(\bar{g}; \Phi, \rho)$  for the given  $\mathcal{C}^\infty$  initial data.

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