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ABSENCE OF NON-COMMUTATIVE MATRIX OBSERVABLES FOR Q-STATE POTTS MODELS

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ABSTRACT. This article is an expository work introducing the subject of lattice models in statistical physics and the types of observables that can be used to prove convergence, as well as a proof for the q-state Potts model showing that non-commutative matrix observables do not exist.

RÉSUMÉ. Cet article est une introduction au sujet des modèles sur réseau en physique statistique et les types d'observables qui peuvent être utilisées pour démontrer la convergence, et aussi une démonstration qu'il n'existe pas d'observable matricielle non-commutative pour le modèle "q-state Potts".

1. **Introduction** The study of convergence of random walks to well defined curves is founded in the fields of complex analysis, probability theory, physics and combinatorics. The foundations of this subject were motivated by physicists interested in the properties of one-dimensional models that represented some form of physical phenomenon. One of the classical models in statistical mechanics is that of the *Ferromagnetic Ising model* (denoted as *Ising model* for short) which can be used to model a magnet acting on a object like iron [1]. Each atom of iron is given a spin up (1) or spin down (-1) depending on the magnet's effect [4]. In 1943, physicists Ashkin and Teller developed their 4-state model, commonly known as the Ashkin-Teller model, based upon similar physical properties. The Ashkin-Teller model was motivated by the four different spin/energy interactions of the neighbouring atoms on the lattice [1]. In 1952, the Ashkin-Teller model and the Ising model were generalized to the q-state Potts model by R.B. Potts. The q-state model considers the case where there are q-states instead of four. The case where $0 \leq q \leq 4$ is denoted as dense and the case where $q > 4$ is denoted as dilute [4]. These models allow physicists and mathematicians to study properties regarding order disorder transitions, their associated critical points and whether certain operators can be scaled or examined universally.

By taking physical models and generalizing them into abstract mathematical terms, macroscopic properties about the model could be determined from the microscopic level [1]. One such example was the convergence result of the simple

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random walk to Brownian motion. Oded Schramm, expanded on this work and developed the process of Stochastic Loewner Evolution of parameter κ (denoted SLE(κ)). The motivation of Schramm in creating such a function was based upon studying planar Brownian motion, loop-erased random walks, as well as other walks on a discrete lattice (i.e., hexagonal, square, triangular, etc.).

Following the work of Schramm, researchers such as Stanislav Smirnov, Dmitry Chelkak and Hugo Duminil-Copin examined properties of different physical models such as that of Ising and Potts and developed theories and conjectures relating them to the SLE [2]. They used calculable expressions, known as observables, to prove that certain properties were conformally invariant within the domain. Through the use of a special type of observable known as the (spin-) fermionic observable, the convergence of the law of interfaces of the Ising and FK-Ising models to their respective SLEs was proven. Together, Smirnov and Copin, provided a rigorous formulation of a physics question with a mathematically rigorous solution through the following two theorems:

Theorem 1.1. *The law of interfaces of the critical Ising model converges in the scaling limit to a conformally invariant limit described by the Schramm-Loewner Evolution of parameter* $\kappa = 3$. *[4]*

Theorem 1.2. *The law of interfaces of the critical FK-Ising model converges in the scaling limit to a conformally invariant limit described by the Schramm-Loewner Evolution of parameter* $\kappa = \frac{16}{3}$. *[4]*

In addition to the fermionic observable, Copin and Smirnov introduced the parafermionic observable which he used to prove that the connective constant of the honeycomb lattice was equal to $\sqrt{2+\sqrt{2}}$ [3]. The parafermionic observable is similar to the fermionic observable except that it is defined over a different set and is based upon the winding of its respective curve. Other types of observables exist which can be defined in terms of the geometry of the lattice, the length of the walk on the lattice, the winding of the walk, crossing probabilities of the curve and other properties. The goal in using such observables is to prove that they converge to some conformally invariant object. Smirnov made a conjecture, similar to the above theorems, which presented a more general relation between the q-state Potts model and SLE in [5].

Conjecture. *For all* $q \in [0,4]$*, as the lattice step goes to zero, the law of the interface converges to Schramm-Loewner evolution with* $\kappa = \frac{4\pi}{arccos(\frac{-\sqrt{q}}{2})}$. *[5]*

In [4], Copin and Smirnov relied heavily on the convergence of the commutative observables for the interfaces of the critical Ising model and critical FK-Ising model based on local relations. If a local non-commutative fermionic or parafermionic observable were found for the q-state Potts model, it may be possible to prove the conjecture in a similar manner as the proofs for the above theorems. It is not yet known whether or not the proofs would work for $q \neq 2$ when the observables are non-commutative. If a local non-commutative observable can be introduced, all that is required is to show that it converges to a conformally invariant object. This article will show that no such local non-commutative observables exists.

THEOREM 1.3. *Every matrix valued (or, more generally, ring-valued) local observable for a q-state Potts model, on the square lattice, is commutative.*

2. Models The definitions of the models in this section will follow the same setup as Peres' notes in [6]. To define the Ising model, let $G = (V, E)$ be a finite subgraph of the regular square lattice $\mathbb{L} \subset \mathbb{R}^2$, where V are the vertices and E are the edges, and $\beta \in \mathbb{R} > 0$. Then the Ising model is a probability space over all configurations $\sigma : V \to \{-1, 1\}$ with Gibbs measure

$$\mu_\beta(\sigma) = \frac{e^{\beta \sum_{x \sim y} \sigma(x)\sigma(y)}}{Z} \tag{2.1}$$

where $Z = \sum_\sigma e^{\beta \sum_{x\sim y} \sigma(x)\sigma(y)}$ is the normalizing factor.

Furthermore, the Ising model can be defined on a simply connected subset Ω of $\mathbb{Z}^2$ with the *Dobrushin boundary conditions*, where a and b are fixed points on $\partial\Omega$. The counterclockwise segment along the boundary $\partial\mathbb{L}$ from vertices a_δ and b_δ (the vertices on the graph of step size δ closest to a and b) are set with spin -1 and the clockwise segment along the boundary is set spin 1. The FK-Ising model, as discussed in the introduction, is an extension of the Ising model. It is the Ising model coupled with the FK percolation (defined by 2.2), where $q = 2$. This formulation of the FK-Ising model is essential for the definition of the FK-Ising fermionic observable discussed in the next section.

Similar to the Ising model, the *Random Cluster Model* is also defined on a graph $G = (V, E)$ yet the configuration used is with respect to the edges and not the vertices. Consider an edge configuration $\eta : E \to \{0, 1\}$ where each edge from vertex to vertex is given either the value 0 or 1. Denote $k(\eta)$ as the number of connected components in the subgraph of G containing all vertices and all edges e such that $\eta(e) = 1$. Then the Random Cluster Model is the probability space with parameters $p \in [0, 1]$ and $q > 1$ over all edge configurations $\eta : E \to \{0, 1\}$ with measure

$$\mu_{p,q}(\eta) = \frac{p^{\sum_e \eta(e)}(1-p)^{\sum_e (1-\eta(e))} q^{k(\eta)}}{Z_{p,q}} \tag{2.2}$$

where $Z_{p,q}$ is the appropriate normalizing factor. As with the Ising model, it is possible to introduce boundary conditions, such that for $q \in \mathbb{Z} > 0$, $B \subset V$ and $\nu : B \to \{0, 1\}$ where ν is a configuration assignment of boundary vertices and where $\tilde{k}(\eta)$ is the number of components in η that do not contain any vertex of B. By adding further restrictions, different measures on the lattice structure can be defined and relations between the measures can be proven [4].

In addition to the cluster model, the Ising model can be generalized to the *q-state Potts model.* This model is the probability space over all configurations

$\sigma : V \to \{1, 2, ..., q\}$, for some integer $q \geq 2$, with the Gibbs free measure

$$\mu_{\beta,q}(\sigma) = \frac{e^{-2\beta \sum_{x\sim y} 1_{\sigma(x)\neq\sigma(y)}}}{\tilde{Z}} \tag{2.3}$$

where $\tilde{Z}$ is the appropriate normalizing factor. Every particle at its respective vertex in the graph G can have any spin from $1, 2, ..., q$ rather than the two ($\{0, 1\}$). As with the other models, the definition of the observables in the q-state Potts model is influenced by the different boundary conditions (i.e., free, Dobrushin, etc.) that are set. The q-state Potts model with $q = 4$ and general $q \geq 0$ will be examined in section 4 with the non-existence proofs.

3. **Observables** Observables are calculable values of a particular model and can be used to prove convergence results. For the FK-Ising model with Dobrushin boundary conditions and parameter p, the edge fermionic observable is defined on the edges of the lattice domain Ω by the expression:

$$F_{\Omega,a_\delta,b_\delta,p}(e) = \mathbb{E}^{a_\delta,b_\delta}_{\Omega,p}[e^{\frac{iW_\gamma(e,b_\delta)}{2}} 1_{e\in\gamma}]. \tag{3.1}$$

where γ is the exploration path between a_δ and b_δ with expectation taken with respect to the paths.

In addition to the fermionic observable, Smirnov introduced the notion of the parafermionic observable in [3] to prove that the connective constant of the honeycomb lattice was $\sqrt{2+\sqrt{2}}$. While it has similarities to the fermionic observable, the parafermionic observable is defined on a self-avoiding walk γ on midedges Ω_M of the domain. The term $W_\gamma(a, b)$ is the winding of the curve γ from midedge a to b and $l(\gamma)$ is the length of γ:

$$F(z) = F(a, z, x, \sigma) = \sum_{\gamma\subset\Omega_M : a\to z} e^{-i\sigma W_\gamma(a,z)} x^{l(\gamma)}. \tag{3.2}$$

The fermionic observable described above was used with conformal invariance theory to prove that the law of interfaces in the FK-Ising model with Dobrushin boundary conditions converges to the Schramm-Loewner Evolution with parameter $\kappa = \frac{16}{3}$ [2]. This result was based upon counting the local equivalent contributions of different configurations around vertices and finding scalar quantities that represented the contributions of the curve. To determine that the value of the winding was relevant to the Ising model, a contribution table was set (see table 1) between two equivalent curves. One curve (ω) touched two edges with no loop while the other curve ($s(\omega)$) touched four edges with a loop (represented graphically in Figure 1). Both curves have the same contribution value where λ is a weight parameter and W_ω is a dummy variable.

Observe that by dropping the W_ω notation and setting the values of the contributions equal to each other, the following relation is obtained:

$$1 + \lambda = \frac{\lambda}{\sqrt{2}} + \frac{\lambda^2}{\sqrt{2}} + \frac{\lambda^{-1}}{\sqrt{2}} + \frac{\lambda}{\sqrt{2}}. \tag{3.3}$$

Table 1: Contributions in Ising model with unknown parameter λ

configuration	W	E	N	S
ω	W_ω	0	0	λW_ω
$s(\omega)$	$W_\omega/\sqrt{2}$	$\lambda^2 W_\omega/\sqrt{2}$	$\lambda^{-1} W_\omega/\sqrt{2}$	$\lambda W_\omega/\sqrt{2}$

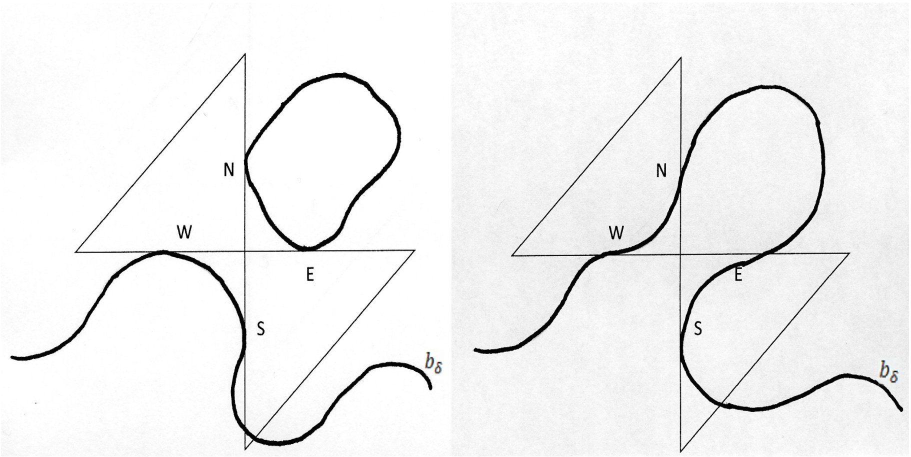

Figure 1: Two configurations ω and $s(\omega)$ where one has an exploratory path without a loop and the other with a loop, respectively

Solving the cubic yields the solutions $\lambda = -1, e^{\frac{i\pi}{4}}, e^{\frac{-i\pi}{4}}$ where the value of $e^{\frac{i\pi}{4}}$ represents the winding of the curve. This technique in solving for the unknown parameter λ can be generalized to matrices and a free ring structure with respect to the q-state Potts model. The notation used for the local relations and their respective observables is defined and will be used throughout the remainder of the article.

DEFINITION 3.1. The q-state *Smirnov relations* are the local relations in the q-state Potts model determined by the contributions of the turns in the curve

DEFINITION 3.2. The *Smirnov-type local observable* is the local commutative observable in the q-state Potts model on the square lattice that satisfies its respective q-state Smirnov relation.

4. **Absence of Non-commutative Observables** In Table 1, the contributions for the Ising model were normalized by a factor of $\sqrt{2}$, as there were only two states, and the value of each turn was represented by a factor of λ. Here, the 4-state Potts model is examined where the choice of λ is replaced with matrix

M and λ^{-1} with matrix N and the normalizing factor of $\sqrt{2}$ is replaced with normalizing factor 2. Table 2 presents the different contributions of the curve at each of the edges.

Table 2: Contributions in 4-state Potts model with unknown matrix parameters M and N

configuration	W	E	N	S
ω	W_ω	0	0	MW_ω
$s(\omega)$	$W_\omega/2$	$M^2W_\omega/2$	$NW_\omega/2$	$MW_\omega/2$

By calcuating a similar equation to that of 3.3, the Smirnov relation between the matrices M and N is obtained, where I is the identity matrix with respect to the dimension of M and N.

$$N = I - M(M - I) \tag{4.1}$$

The values of the matrix contributions in the 4-state Potts model can be furthered generalized to the q-state Potts model by changing the normalizing factor to $\sqrt{q}$ as shown in Table 3. From the value of each edge's contribution, the q-state Smirnov relation is determined where I is the identity matrix with respect to the dimension of M and N:

$$N = -M^2 + M(\sqrt{q}I - I) + (\sqrt{q}I - I). \tag{4.2}$$

Table 3: Contributions in the q-state Potts model with unknown matrix parameters M and N

configuration	W	E	N	S
ω	W_ω	0	0	MW_ω
$s(\omega)$	$W_\omega/\sqrt{q}$	$M^2W_\omega/\sqrt{q}$	$NW_\omega/\sqrt{q}$	$MW_\omega/\sqrt{q}$

By using this relation 4.2, it can be shown that there are no finite dimensional non-commutative matrices M and N that solve the equation.

Theorem 4.1. *No Smirnov-type local matrix observables exist.*

The proof of the theorem will proceed by induction on the dimension of the square matrices M and N. For every choice of matrix M, the matrix N is calculated from Equation 4.2. The base cases of dimension 2 and dimension 3 will be proven and then the general case of dimension n.

PROOF. The equation that determines the contribution of the two matrices is given in 4.2. Since every matrix can be written as a product of a matrix of eigenvectors and a Jordan block matrix (i.e., $M = T^{-1}AT$), then it is sufficient to only consider the cases where M is in Jordan form as N can be solved directly. For the 2x2 matrices, there are two cases to consider: (i) M is diagonal and (ii) M is a dimension 2 Jordan block. Observe that the set of diagonal matrices is closed under multiplication and addition, which implies $N = -M^2 + M(\sqrt{q} - I) + (\sqrt{q} - I)$ is a diagonal matrix. Therefore, M and N will commute. In the case where M has the following form:

$$M = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

then N is directly computed to be:

$$N = \begin{pmatrix} \lambda^2 + \lambda(\sqrt{q} - 1) + (\sqrt{q} - 1) & -2\lambda + (\sqrt{q} - 1) \\ 0 & \lambda^2 + \lambda(\sqrt{q} - 1) + (\sqrt{q} - 1) \end{pmatrix}.$$

It is a simple calculation to show that M and N commute:

$$MN = NM = \begin{pmatrix} \lambda(\lambda^2 + \lambda(\sqrt{q} - 1) + (\sqrt{q} - 1)) & \lambda^2 + \lambda(\sqrt{q} - 1) + (\sqrt{q} - 1) + (-2\lambda + (\sqrt{q} - 1))\lambda \\ 0 & \lambda(\lambda^2 + \lambda(\sqrt{q} - 1) + (\sqrt{q} - 1)) \end{pmatrix}.$$

For the dimension 3 matrix N, the case when M is made up of a dimension 2 Jordan block and a third diagonal element is disregarded, as the respective solution blocks of matrix N were already shown to commute in the dimension 2 case. Thus, the only case to consider is where M is a full Jordan block:

$$M = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

in which N is computed to be:

$$N = \begin{pmatrix} \lambda^2 + \lambda(\sqrt{q} - 1) + (\sqrt{q} - 1) & -2\lambda + (\sqrt{q} - 1) & -1 \\ 0 & \lambda^2 + \lambda(\sqrt{q} - 1) + (\sqrt{q} - 1) & -2\lambda + (\sqrt{q} - 1) \\ 0 & 0 & \lambda^2 + \lambda(\sqrt{q} - 1) + (\sqrt{q} - 1) \end{pmatrix}.$$

Observe that since M is in full Jordan block form, then N will always be a three-band matrix with the same values as in the dimension 3 case for all higher finite dimensions. This will be used to show commutativity in the induction step. Once again, in the dimension 3 case, the matrices M and N are shown to commute by calculating their product.

Assume by strong induction that for all matrices M and N, of dimension $\leq n-1$ that satisfy the contribution equation for the 4-state Potts model, commute. Suppose M is a Jordan block of dimension n. Then the only case to consider is where the eigenvalue of M has geometric multiplicity n and algebraic multiplicity 1. All other cases can be disregarded as the smaller Jordan blocks within M will automatically commute with their respective calculated blocks in N by the induction hypothesis. Thus, M has the following form:

$$M = \begin{pmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

which as discussed in the dimension 3 case will yield a 3 band matrix N as follows:

$$N = \begin{pmatrix} a & b & -1 & 0 & \dots & 0 \\ 0 & a & b & -1 & 0 & \dots \\ 0 & 0 & a & b & -1 & 0\dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a & b & -1 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}$$

where $a = \lambda^2 + \lambda(\sqrt{q} - 1) + (\sqrt{q} - 1)$ and $b = -2\lambda + (\sqrt{q} - 1)$. To show commutativity of matrices M and N, observe that for the product MN, the representation of the row of M and the column of N where $j \geq 4$ have the forms:

$$\begin{array}{rcccccccccc} & 1^{st} & 2^{nd} & ... & i^{th}-1 & i^{th} & i^{th}+1 & i^{th}+2 & i^{th}+3 & & n \\ \text{Row of N:} & (\ 0 & 0 & \dots & 0 & a & b & -1 & 0 & \dots & 0\ ) \end{array}$$

$$\begin{array}{rccccccccc} & 1^{st} & 2^{nd} & ... & j^{th}-2 & j^{th}-1 & j^{th} & j^{th}+1 & & n \\ \text{Column of M:} & (\ 0 & 0 & \dots & 0 & 1 & \lambda & 0 & \dots & 0\ ) \end{array}$$

and for the product of NM, they have a similar representation as shown below.

$$\begin{array}{rccccccccc} & 1^{st} & 2^{nd} & ... & i^{th}-1 & i^{th} & i^{th}+1 & i^{th}+2 & & n \\ \text{Row of M:} & (\ 0 & 0 & \dots & 0 & \lambda & 1 & 0 & \dots & 0\ ) \end{array}$$

$$\begin{array}{rcccccccccc} & 1^{st} & 2^{nd} & ... & j^{th}-3 & j^{th}-2 & j^{th}-1 & j^{th} & j^{th}+1 & & n \\ \text{Column of N:} & (\ 0 & 0 & \dots & 0 & -1 & b & a & 0 & \dots & 0\ ) \end{array}$$

Then it is clear that when $i > j$ or $i < 3$ the dot product of these vectors are both equal to zero. For the cases where $i = j$, $i = j-1$, $i = j-2$ and $i = j-3$, the dot products of these vectors are equal. Therefore, in the n dimensional case, the matrices M and N that satisfy the contribution Equation 4.2 will commute. ☐

THEOREM 4.2. *Let $< M, N, I >$ be a free ring structure with multiplication operation $\cdot$, addition operation $+$, identity element I and generators M and N. Then there exists no Smirnov-type local free ring observables with respect to the product operation of the ring.*

PROOF. Consider the factor ring $< M, N, I > /(N = -M^2 + M(\sqrt{q} - I) + (\sqrt{q} - I))$ then observe the products MN and NM. They are as follows:

$$MN = M(-M^2 + M(\sqrt{q} - I) + (\sqrt{q} - I) = -M^3 + M^2(\sqrt{q} - I) + M(\sqrt{q} - I)$$

$$NM = (-M^2 + M(\sqrt{q} - I) + (\sqrt{q} - I))M = -M^3 + M(\sqrt{q} - I)M + (\sqrt{q} - I)M.$$

Then by the distributivity, and identity element properties, it is clear that $MN = NM$ and thus the factor ring elements M and N commute. ☐

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