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Comments Related to Infinite Wedge Representations

Nathan Grieve

Presented by George A. Elliott, FRSC

Abstract. We study the infinite wedge representation and show how it is related to the universal central extension of $g[t, t^{-1}]$, the loop algebra of a complex semi-simple Lie algebra g. We also give an elementary proof of the boson-fermion correspondence. Our approach to proving this result is based on a combinatorial construction combined with an application of the Murnaghan-Nakayama rule.

Résumé. Nous étudions l'algèbre extérieure en dimension infinie et montrons comment elle est reliée à l'extension centrale universelle de $g[t,t^{-1}]$, l'algèbre de lacets sur une algèbre de Lie g semi-simple complexe. De plus, nous donnons une preuve élémentaire de la correspondance boson-fermion. Pour ce faire, nous utilisons une construction combinatoire, ainsi que la règle de Murnaghan-Nakayama.

1. Introduction In this article, we make two remarks about the *infinite wedge representation.* To describe what we do let $\mathbf{gl}(\infty)$ denote the Lie algebra of $\mathbb{Z}\times\mathbb{Z}$ *band infinite matrices.* Then $\mathbf{gl}(\infty)$ consists of those matrices $A = (a_{ij})_{i,j\in\mathbb{Z}}$ with $a_{ij} \in \mathbb{C}$ and $a_{ij} = 0$ for all $|i - j| \gg 0$. Next let $\widehat{\mathbf{gl}}(\infty)$ denote the Lie algebra determined by the 2-cocycle $\mathrm{c}(\cdot,\cdot)$ of $\mathbf{gl}(\infty)$ with values in the trivial $\mathbf{gl}(\infty)$-module $\mathbb{C}$:

$$\mathrm{c}(A, B) := \sum_{i\leqslant 0,k>0} a_{ik}b_{ki} - \sum_{i>0,k\leqslant 0} a_{ik}b_{ki},$$

see for instance [1, p. 12] or [5, p. 115]. The *infinite wedge representation* is a suitably defined, see §4 for precise details, Lie algebra representation $\rho : \widehat{\mathbf{gl}}(\infty) \to \mathrm{End}_{\mathbb{C}}(F)$; here F is the *infinite wedge space* that is the $\mathbb{C}$-vector space determined by the set $\mathscr{S}$ which consists of those ordered strictly decreasing sequences of integers $S = (s_1, s_2, \dots)$, $s_i \in \mathbb{Z}$, with the properties that $s_i = s_{i-1} - 1$ for all $i \gg 0$.

To describe our first theorem, let g be a finite dimensional semi-simple complex Lie algebra, $g[t, t^{-1}]$ its loop algebra and $\widehat{g}$ the universal (central) extension of

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$g[t, t^{-1}]$ in the sense of Garland [3, §2] (see also [13, §7.9]). In Theorem 3.1 we show how the representation ρ is related to $\widehat{g}$.

Our second theorem, Theorem 6.2, gives an elementary proof of the *boson-fermion correspondence*, in the sense of Kac-Raina-Rozhkovskaya [6, Lecture 5, p. 46]. To place this theorem in its proper context, let $\mathfrak{s}$ denote the *oscillator algebra*, which is the universal extension of $\mathbb{C}[t, t^{-1}]$ the loop algebra of the abelian Lie algebra $\mathbb{C}$. The Lie algebra $\mathfrak{s}$ is faithfully represented in $\operatorname{End}_{\mathbb{C}}(F)$ and also in $\operatorname{End}_{\mathbb{C}}(B)$, where B denotes the polynomial ring in countably many variables with coefficients in the ring of Laurent polynomials. The boson-fermion correspondence, as formulated in [6, Lecture 5, p. 46] compare also with [5, §14.9–14.10], concerns extending these representations to all of $\widehat{\mathbf{gl}}(\infty)$ in such a way that an evident $\mathbb{C}$-linear isomorphism $F \to B$ becomes an isomorphism of $\widehat{\mathbf{gl}}(\infty)$-modules; see §6 for more precise details. The traditional approach for proving this result is by way of vertex-operators, see [5] and [6, Lecture 6, p. 46]. The key point to our approach, which does not require the use of vertex-operators, is a combinatorial construction related to partitions, see §5, together with the Murnaghan-Nakayama rule which we recall in §2.6.

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2. Preliminaries

In this section, to fix notation and terminology for what follows, we recall a handful of combinatorial and Lie theoretic concepts. For the most part we use combinatorial terminology and conventions similar to that of [7, I §1 – 5] and Lie theoretic terminology and conventions similar to that of [5, §7 and §14].

2.1. Let $\mathscr{P}$ denote the set of partitions. Then $\mathscr{P}$ consists of those infinite weakly decreasing sequences of non-negative integers of the form $\lambda = (\lambda_1, \lambda_2, \dots)$ with the property that at most finitely many of the λ_i are nonzero. If $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P}$, then the number $\operatorname{weight}(\lambda) := \sum_{i=1}^{\infty} \lambda_i$ is called the *weight of* λ and we denote by $\mathscr{P}_d$ the set of partitions of weight d. If $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P}$, then we often identify λ with the finite weakly decreasing sequence $(\lambda_1, \dots, \lambda_r)$, where $r = \operatorname{length}(\lambda) := \max\{i : \lambda_i \neq 0\}$.

2.2. The *Young diagram* of a partition $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P}$ is defined to be the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \leqslant j \leqslant \lambda_i$. When drawing the Young diagram associated to a partition we use the convention that the first coordinate is the row index, starts at 1 and increases from left to right. Similarly, the second coordinate is the column index, starts at 1 and increases downward. We refer to the elements (i, j) of a Young diagram as the *boxes* of the associated partition and the entires i and j as the sides of the box.

2.3. If $\lambda, \mu \in \mathscr{P}$ and $\lambda \supseteq \mu$, so $\lambda_i \geqslant \mu_i$ for all $i \geqslant 1$, then the set theoretic difference of the Young diagrams corresponding to λ and μ is denoted by $\lambda \setminus \mu$ and is called a *skew diagram.* If $\lambda, \mu \in \mathscr{P}$ and $\lambda \supseteq \mu$, then let $\theta := \lambda \setminus \mu$ denote the skew diagram that they determine. By a *path* in θ, we mean a sequence $x_0, x_1, \dots, x_m$ with $x_i \in \theta$, such that x_{i-1} and x_i have a common side for $1 \leqslant i \leqslant m$. A subset $\nu \subseteq \theta$ is said to be *connected* if every two boxes in ν can be connected by a path in ν. The *length* of θ is defined to be the number of boxes that appear in its diagram and is denoted by $\#\theta$. We say that θ is a *border strip* if it is connected and if it contains no 2×2 box. Finally, if θ is a border strip, then we denote its *height* by $\operatorname{height}(\theta)$ and define it to be one less then the number of rows that it occupies.

2.4. The symmetric group S_n acts on the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ by permuting the variables and we let Λ_n denote the subring of invariants. We then have that $\Lambda_n = \bigoplus_{k \geqslant 0} \Lambda_n^k$ where $\Lambda_n^k \subseteq \Lambda_n$ is the subspace of symmetric polynomials of degree k. If $k \in \mathbb{Z}_{\geqslant 0}$, $m, n \in \mathbb{Z}_{\geqslant 1}$ and $m \geqslant n$, we have evident restriction maps $\rho^k_{m,n} : \Lambda^k_m \to \Lambda^k_n$; let $\Lambda^k = \varprojlim \Lambda^k_n$ and $\Lambda = \bigoplus_{k \geqslant 0} \Lambda^k$. Then $\Lambda = \mathbb{C}[h_1, h_2, \dots]$ where the h_k are such that their image in Λ^k_n is the kth complete symmetric function in the variables $x_1, \dots, x_n$.

2.5. Let $H(Z) := \sum_{k \geqslant 0} h_k Z^k \in \Lambda[[Z]]$ and define $p_k \in \Lambda$ by the coefficient of Z^{k-1} in the power series $P(Z) := H'(Z)/H(Z)$. The image of each p_k in Λ^k_n is the kth power sum in the variables $x_1, \dots, x_n$ and the h_k can be expressed in terms of the p_k via the equality of power series $H(Z) = \exp\left(\sum_{k \geqslant 1} t_k Z^k\right)$; here $t_k = p_k/k$. The Schur functions s_λ, defined for all partitions $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P}$, are defined by $s_\lambda := \det\left(h_{\lambda_i - i + j}\right)_{1 \leqslant i,j \leqslant n}$, where $n = \operatorname{length}(\lambda)$, and form a $\mathbb{C}$-basis for Λ. In what follows we let $\langle \cdot, \cdot \rangle$ denote the symmetric bilinear form on Λ for which the Schur polynomials are orthonormal. In particular, $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$.

2.6. By abuse of notation, we let $p_k \in \operatorname{End}_{\mathbb{C}}(\Lambda)$ be the $\mathbb{C}$-linear endomorphism given by multiplication by p_k. The adjoint of p_k with respect to $\langle \cdot, \cdot \rangle$, which we denote by $p_k^\perp$, is the $\mathbb{C}$-linear endomorphism given by the differential operator $k\frac{\partial}{\partial p_k}$, [7, p. 76].

The effect of the operator p_k in the basis of Schur polynomials is given by the Murnaghan-Nakayama rule:

$$p_k s_\lambda = \sum_{\substack{\nu \supseteq \lambda, \\ \nu \setminus \lambda \text{ is a border strip} \\ \text{of length } k}} (-1)^{\operatorname{height}(\nu \setminus \lambda)} s_\nu, \tag{2.1}$$

[10, p. 601]. Using (2.1), in conjunction with [7, I.V. Ex. 3, p. 75], we deduce the adjoint form of the Murnaghan-Nakayama rule:

$$p_k^\perp s_\lambda = \sum_{\substack{\lambda \supseteq \nu, \\ \lambda \setminus \nu \text{ is a border strip} \\ \text{of length } k}} (-1)^{\operatorname{height}(\lambda \setminus \nu)} s_\nu. \tag{2.2}$$

2.7. We let $\mathrm{Mat}(\infty)$ denote the $\mathbb{C}$-vector space of $\mathbb{Z}\times\mathbb{Z}$ matrices with entries in $\mathbb{C}$. If $A=(a_{ij})_{i,j\in\mathbb{Z}}, B=(b_{ij})_{i,j\in\mathbb{Z}}\in\mathrm{Mat}(\infty)$ and $a_{ik}b_{kj}=0$, for all $i,j\in\mathbb{Z}$ and almost all $k\in\mathbb{Z}$, then their product is given by $C=AB:=(c_{ij})_{i,j\in\mathbb{Z}}$, where $c_{ij}=\sum_{k\in\mathbb{Z}}a_{ik}b_{kj}$. Further, we let E_{ij} denote the element of $\mathrm{Mat}(\infty)$ with i,j entry equal to 1 and all other entries equal to zero. We say that a matrix $A=(a_{ij})_{i,j\in\mathbb{Z}}\in\mathrm{Mat}(\infty)$ is a *band infinite matrix* if $a_{ij}=0$ for all $|i-j|\gg 0$. We denote the collection of band infinite matrices by $\mathbf{gl}(\infty)$ and regard it as a Lie algebra with Lie bracket given by $[A,B]=AB-BA$. We often express elements of $\mathbf{gl}(\infty)$ as infinite sums of matrices. For example, the identity matrix $1_{\mathbb{Z}\times\mathbb{Z}}=(\delta_{ij})_{i,j\in\mathbb{Z}}$, can be expressed as $1_{\mathbb{Z}\times\mathbb{Z}}=\sum_{p\in\mathbb{Z}}\mathrm{E}_{pp}$. Also every element of $\mathbf{gl}(\infty)$ can be written as a finite linear combination of matrices of the form $\sum_i a_i\mathrm{E}_{i,i+k}$, where $k\in\mathbb{Z}$ and $a_i\in\mathbb{C}$.

2.8. Let $\mathfrak{gl}_N[t,t^{-1}]:=\mathbb{C}[t,t^{-1}]\otimes_{\mathbb{C}}\mathfrak{gl}_N(\mathbb{C})$ which we regard as a Lie algebra with Lie bracket determined by

$$[f(t)\otimes A, g(t)\otimes B]=f(t)g(t)\otimes[A,B].$$

If $t^m\otimes e_{ij}$, for $i,j=1,\dots,N$ and $m\in\mathbb{Z}$, denotes the standard basis elements of $\mathfrak{gl}_N[t,t^{-1}]$, we then have that

$$[t^m\otimes e_{ij}, t^n\otimes e_{k\ell}]=t^{m+n}\otimes(\delta_{jk}e_{i\ell}-\delta_{\ell i}e_{kj}),$$

and that the map

$$\iota_N:\mathfrak{gl}_N[t,t^{-1}]\to\mathbf{gl}(\infty), \tag{2.3}$$

determined by

$$t^m\otimes e_{ij}\mapsto\sum_{k\in\mathbb{Z}}\mathrm{E}_{N(k-m)+i,Nk+j},$$

is a monomorphism of Lie algebras. The image of ι_N is the Lie algebra of *N-periodic band infinite matrices*, that is those $\mathbb{Z}\times\mathbb{Z}$ band infinite matrices $A=(a_{ij})_{i,j\in\mathbb{Z}}$ for which $a_{i+N,j+N}=a_{ij}$, for all $i,j\in\mathbb{Z}$.

2.9. To define the Lie algebra $\widehat{\mathbf{gl}}(\infty)$, first let

$$J:=\sum_{m\leqslant 0}\mathrm{E}_{mm}-\sum_{m>0}\mathrm{E}_{mm}\in\mathbf{gl}(\infty)$$

and observe that if A and B are elements of $\mathbf{gl}(\infty)$, then the matrix $[J,A]B$ has at most finitely many nonzero diagonal elements and the expression $\frac{1}{2}\operatorname{tr}([J,A]B)$ is a well defined element of $\mathbb{C}$. In particular, we have

$$\frac{1}{2}\operatorname{tr}([J,A]B)=\sum_{i\leqslant 0,k>0}a_{ik}b_{ki}-\sum_{i>0,k\leqslant 0}a_{ik}b_{ki}, \tag{2.4}$$

and we define the Lie algebra $\widehat{\mathbf{gl}}(\infty)$ to be the central extension determined by the following 2-cocycle of $\mathbf{gl}(\infty)$ with values in the trivial $\mathbf{gl}(\infty)$-module $\mathbb{C}$:

$$\mathrm{c}(A,B) := \frac{1}{2}\operatorname{tr}([J,A]B) = \sum_{i\leqslant 0, k>0} a_{ik}b_{ki} - \sum_{i>0, k\leqslant 0} a_{ik}b_{ki}. \tag{2.5}$$

As a special case of (2.5), we have that

$$\mathrm{c}(\mathrm{E}_{ij},\mathrm{E}_{k\ell}) = \begin{cases} -1 & i=\ell>0,\ j=k\leqslant 0 \\ 1 & i=\ell\leqslant 0,\ j=k>0 \\ 0 & \text{otherwise,} \end{cases} \tag{2.6}$$

for $i,j,k,\ell \in \mathbb{Z}$; compare with [1, p. 12] or [5, p. 115 and p. 313]. Explicitly, as a $\mathbb{C}$-vector space

$$\widehat{\mathbf{gl}}(\infty) = \mathbb{C} \oplus \mathbf{gl}(\infty),$$

and the Lie bracket is defined by

$$[(a,x),(b,y)] = (\mathrm{c}(x,y),[x,y]),$$

for all $(a,x),(b,y) \in \widehat{\mathbf{gl}}(\infty)$.

2.10. We regard $R := \mathbb{C}[t,t^{-1}]$, the ring of Laurent polynomials, as the loop algebra of the abelian Lie algebra $\mathbb{C}$. The *oscillator algebra* is the Lie algebra $\mathfrak{s}$ determined by the 2-cocycle with values in the trivial R-module $\mathbb{C}$ given by:

$$\omega : \mathbb{C}[t,t^{-1}] \times \mathbb{C}[t,t^{-1}] \to \mathbb{C},$$

$$\omega\left((f(t),g(t)\right) := \operatorname{res}\left(\frac{df}{dt}g\right).$$

Concretely,

$$\mathfrak{s} = \mathbb{C} \oplus \mathbb{C}[t,t^{-1}],$$

and the bracket is given by

$$[(a,t^m),(b,t^n)] = (m\delta_{m,-n},0),$$

for all $a,b \in \mathbb{C}$ and $m,n \in \mathbb{Z}$.

2.11. As in [5, p. 313], we realize the oscillator algebra $\mathfrak{s}$ as a subalgebra of $\widehat{\mathbf{gl}}(\infty)$ by the monomorphism of Lie algebras

$$\delta_0 : \mathfrak{s} \to \widehat{\mathbf{gl}}(\infty), \tag{2.7}$$

defined by

$$(a,t^m) \mapsto \left(a, \sum_{j\in\mathbb{Z}} \mathrm{E}_{j,j+m}\right).$$

3. The Lie Algebra $\widehat{\mathbf{gl}}(\infty)$ and Universal Extensions In this section we establish Theorem 3.1 which shows how the Lie algebra $\widehat{\mathbf{gl}}(\infty)$ is related to the Lie algebra $\widehat{g}$ which we define to be the *universal (central) extension* of $g[t,t^{-1}]$ the Loop algebra of g a complex finite dimensional semi-simple Lie algebra.

3.1. Let g be a complex finite dimensional semi-simple Lie algebra and $\kappa(\cdot,\cdot)$ its killing form. We denote by $\widehat{g}$ the *universal extension* of $g[t,t^{-1}]$. Then $\widehat{g}$ is the central extension determined by the 2-cocycle

$$u(\cdot,\cdot) : g[t,t^{-1}] \times g[t,t^{-1}] \to \mathbb{C} \tag{3.1}$$

defined by

$$u\left(\sum t^i \otimes x_i, \sum t^j \otimes y_j\right) := \sum i\kappa(x_i, y_{-i}), \tag{3.2}$$

[3, §2] see also [13, §7.9] especially [13, §7.9.6, p. 250].

To relate $\widehat{g}$ and $\widehat{\mathbf{gl}}(\infty)$, we choose a basis for g and then consider its *extended adjoint representation*:

$$1 \otimes \mathrm{ad} : g[t,t^{-1}] \to \mathbf{gl}(\infty), \tag{3.3}$$

see (3.6) below.

The morphism $1 \otimes \mathrm{ad}$, given by (3.3), allows us to compare the pullback of $\widehat{\mathbf{gl}}(\infty)$, with resect to $1 \otimes \mathrm{ad}$, and the universal extension $\widehat{g}$. In §3.3, we prove:

THEOREM 3.1. *The universal central extension of $g[t,t^{-1}]$ is the pull-back of $\widehat{\mathbf{gl}}(\infty)$ via $1 \otimes \mathrm{ad}$, the extended adjoint representation of g.*

3.2. Before proving Theorem 3.1 we first observe:

PROPOSITION 3.2. *The pullback of $\mathrm{c}(\cdot,\cdot)$ to $\mathfrak{gl}_N[t,t^{-1}]$ via ι_N is given by:*

$$\mathrm{c}(\iota_N(t^m \otimes x), \iota_N(t^n \otimes y)) = m\delta_{m,-n}\,\mathrm{tr}(xy). \tag{3.4}$$

PROOF. In light of the map (2.3), it suffices to check that, for fixed $N \in \mathbb{Z}_{\geqslant 1}$, $1 \leqslant i,j,k,\ell \leqslant N$, $m,n \in \mathbb{Z}$, we have

$$\mathrm{c}\left(\sum_{p\in\mathbb{Z}} \mathrm{E}_{N(p-m)+i,Np+j}, \sum_{q\in\mathbb{Z}} \mathrm{E}_{N(q-n)+k,Nq+\ell}\right) = \begin{cases} m & \text{if } j=k,\ i=\ell \text{ and } m=-n. \\ 0 & \text{otherwise.} \end{cases}$$

To compute

$$\mathrm{c}\left(\sum_{p\in\mathbb{Z}} \mathrm{E}_{N(p-m)+i,Np+j}, \sum_{q\in\mathbb{Z}} \mathrm{E}_{N(q-n)+k,Nq+\ell}\right),$$

considering (2.5), it is clear that we need to understand the quantity:

$$\sum_{\substack{p,q\in\mathbb{Z},\\ Np+j>0,\\ k+N(q-n)>0,\\ i+N(p-m)\leqslant 0,\\ Nq+\ell\leqslant 0}} \delta_{Np+j,N(q-n)+k}\delta_{N(p-m)+i,Nq+\ell} - \sum_{\substack{p,q\in\mathbb{Z},\\ Np+j\leqslant 0,\\ k+N(q-n)\leqslant 0,\\ i+N(p-m)>0,\\ Nq+\ell>0}} \delta_{Np+j,N(q-n)+k}\delta_{N(p-m)+i,Nq+\ell}. \tag{3.5}$$

To this end, we make the following deductions:

(a) if (3.5) is nonzero, then $m=-n$;
(b) if $m \geqslant 0$ the first sum appearing in (3.5) is nonzero if and only if $j=k$ and $i=\ell$, while the second sum is zero; the nonzero summands appearing (3.5), when $j=k$ and $i=\ell$, are in bijection with the set of pairs $(p,q)\in\mathbb{Z}\times\mathbb{Z}$ with $-1\leqslant p\leqslant m$ and $n\leqslant q<0$;
(c) if $m<0$, the second sum appearing in (3.5) is nonzero if and only if $j=k$ and $i=\ell$, while the first sum is zero; the nonzero summands appearing in (3.5), when $j=k$ and $i=\ell$, are in bijection with the set of pairs $(p,q)\in\mathbb{Z}\times\mathbb{Z}$ with $m\leqslant p<0$, $0\leqslant q<n$.

The conclusion of Proposition 3.2 follows immediately from these deductions. □

3.3. We now establish Theorem 3.1. To do so, first consider an arbitrary semi-simple Lie algebra g and its adjoint representation

$$\mathrm{ad}: g\to \mathrm{End}_{\mathbb{C}}(g).$$

Let $N=\dim_{\mathbb{C}} g$ and fix a basis for g. By composition we obtain a representation

$$\mathrm{ad}: g\to \mathrm{End}_{\mathbb{C}}(g) \xrightarrow{\sim} \mathfrak{gl}_N(\mathbb{C}),$$

which we can use to define the *extended adjoint representation* of g

$$g[t,t^{-1}] \xrightarrow{1\otimes\mathrm{ad}} \mathfrak{gl}_N(\mathbb{C})[t,t^{-1}] \xrightarrow{\iota_N} \mathbf{gl}(\infty). \tag{3.6}$$

The homomorphism (3.6) allows us to compare the pull-back of $\widehat{\mathbf{gl}}(\infty)$, via $1\otimes\mathrm{ad}$, with $\widehat{g}$.

PROOF OF THEOREM 3.1. It is enough to show that

$$u\left(\sum t^i \otimes x_i, \sum t^j \otimes y_j\right) := \sum i\kappa(x_i, y_{-i})$$

equals

$$\mathrm{c}\left(\sum t^i \otimes \operatorname{ad} x_i, \sum t^j \otimes \operatorname{ad} y_j\right).$$

That this equality holds true follows from the fact that

$$\kappa(x,y) := \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y)$$

and from Proposition 3.2. □

4. Semi-infinite Monomials and the Infinite Wedge Representation

In this section we study certain subsequences of $\mathbb{Z}$ which we refer to as *semi-infinite monomials*, see §4.1. We then describe the *infinite wedge space* and the *infinite wedge representation* of the Lie algebra $\widehat{\mathbf{gl}}(\infty)$, see §4.6 and §4.8 respectively. What we do here is influenced heavily by what is done in [5], [12], [4] and [8]. We give proofs of all assertions for completeness and because they are needed in our proof of Theorem 6.2.

4.1. By a *semi-infinite monomial* we mean an ordered strictly decreasing sequence of integers $S = (s_1, s_2, \dots)$, $s_i \in \mathbb{Z}$, with the properties that $s_i = s_{i-1} - 1$ for all $i \gg 0$. We let $\mathscr{S}$ denote the set of semi-infinite monomials. If $S \in \mathscr{S}$, then define strictly decreasing sequences of integers S_+ and S_- by $S_+ := S \setminus \mathbb{Z}_{\leqslant 0}$ and $S_- := \mathbb{Z}_{\leqslant 0} \setminus S$.

4.2. If $S = (s_1, s_2, \dots) \in \mathscr{S}$, then there exists a unique integer m with the property that $s_i = m - i + 1$ for all $i \gg 0$. We refer to this number as the *charge* of S and denote it by $\operatorname{charge}(S)$, compare with [12, p. 12], [5, p. 310], and [9, A.3] for instance. If $m \in \mathbb{Z}$, then let $\mathscr{S}_m := \{S \in \mathscr{S} : \operatorname{charge}(S) = m\}$. We record the following proposition for later use.

PROPOSITION 4.1. *The following assertions hold true:*

(a) *If $S \in \mathscr{S}$, then* $\operatorname{charge}(S) = \#S_+ - \#S_-$*;*
(b) *Let $m \in \mathbb{Z}$. The map $\lambda : \mathscr{S}_m \to \mathscr{P}$ defined by*

$$S = (s_1, s_2, \dots) \mapsto \lambda(S) = (\lambda_1, \lambda_2, \dots),$$

where

$$\lambda_j := s_j - m + j - 1, \tag{4.1}$$

is a bijection.

Proof. To prove (a), let $S := (s_1, s_2, \dots) \in \mathscr{S}$, and write:

$$S_+ := (s_1, \dots, s_\ell) \text{ and } S_- := (n_1, \dots, n_r); \tag{4.2}$$

here $s_1 > s_2 > \cdots > s_\ell$ and $0 \geqslant n_1 > n_2 > \cdots > n_r$. Considering the definitions of S_+ and S_- we deduce that

$$s_{n+k} = n_r - k \text{ for } n := \ell - n_r - r + 1 \text{ and } k \geqslant 1. \tag{4.3}$$

Now suppose that $m := \ell - r = \#S_+ - \#S_-$ and let $i = n + k$ for $k \geqslant 1$. We then have $m - i + 1 = n_r - k$ which equals s_i by (4.3). Conversely, suppose that $s_i = m - i + 1$ for all $i \gg 0$. We then have for all $k \gg 0$ that

$$s_{n+k} = m - n - k + 1 = m - \ell + n_r + r - k. \tag{4.4}$$

Combining (4.3) and (4.4), we then have

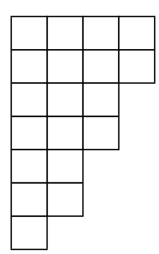
$$n_r - k = m - \ell + n_r + r - k \tag{4.5}$$

and so $m = \ell - r$ as desired.

For (b), first note that the map λ is clearly injective. To see that it is surjective, if $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P}$, then define an element $S = (s_1, s_2, \dots) \in \mathscr{S}_m$ by $s_j = \lambda_j + m - j + 1$. By construction $S \in \mathscr{S}$. To see that $S \in \mathscr{S}_m$ note that $s_j = m - j + 1$ for $j > \text{length}(\lambda)$. $\square$

4.3. Remark. When we express $S \in \mathscr{S}$ as in (4.2), the length of the partition $\lambda(S)$ equals the number n defined in (4.3). Also the weight of the partition $\lambda(S)$ is sometimes referred to as the *energy* of S, [5, p. 310].

4.4. Example. We can use the approach of [11, §7.2] to give a graphical interpretation of Proposition 4.1 for the case $m = 0$. The case $m \neq 0$ can be handled similarly with a shift. As an example, the Young diagram associated to the partition $\lambda = (4, 4, 3, 3, 2, 2, 1) \in \mathscr{P}_{19}$ is:



If we cut this Young diagram along the main diagonal then there are 3 rows in the top piece and 3 columns in the bottom piece. Let u_i, $i = 1, 2, 3$, denote the number of boxes in the i-th row of the top piece and let v_i, $i = 1, 2, 3$, denote the number of boxes in the i-th column of the bottom piece. Then, $u_1 = 3.5$, $u_2 = 2.5$, $u_3 = .5$ and $v_1 = 6.5$, $v_2 = 4.5$, and $v_3 = 1.5$.

If S is the charge zero semi-infinite monomial corresponding to λ, then S is determined by the condition that

$$S_+ = (u_1 + .5, u_2 + .5, u_3 + .5) = (4, 3, 1)$$

and

$$S_- = (-v_3 + .5, -v_2 + .5, -v_1 + .5) = (-1, -4, -6).$$

In other words,

$$S = (4, 3, 1, 0, -2, -3, -5, -7, -8, \dots)$$

is the element of $\mathscr{S}_0$ corresponding to the partition

$$\lambda = (4, 4, 3, 3, 2, 2, 1).$$

We can also relate the set S to the *code*, in the sense of [2, §2], of the partition λ. Specifically, if $n \in \mathbb{Z}$, $n \geqslant 1$ and $n \notin S$, then n corresponds to an R; if $n \geqslant 1$ and $n \in S$, then n corresponds to a U. If $n \in \mathbb{Z}$, $n \leqslant 0$ and $n \in S$, then n corresponds to a U; if $n \leqslant 0$ and $n \notin S$, then n corresponds to an R. The string consisting of these R's and U's is the code corresponding to λ and our set S.

4.5. Let $\lambda : \mathscr{S} \to \mathscr{P}$ denote the extension of the bijections $\lambda : \mathscr{S}_m \to \mathscr{P}$ described in Proposition 4.1 (b). Also, to keep track of various minus signs which appear in what follows, we make the following definition: if $S \in \mathscr{S}$ and $j \in \mathbb{Z}$, then define $\operatorname{count}(j, S)$ to be the number of elements of S that are strictly greater than j, that is:

$$\operatorname{count}(j, S) := \#\{s \in S : j < s\}. \tag{4.6}$$

4.6. The *infinite wedge space* is the $\mathbb{C}$-vector space $F := \bigoplus_{S \in \mathscr{S}} \mathbb{C}$ determined by the set $\mathscr{S}$, see for instance [5, §14.15] or [9, p. 76]. In particular,

$$F = \operatorname{span}_{\mathbb{C}}\{v_S : S \in \mathscr{S}\}$$

where $v_S = (r_T)_{T \in \mathscr{S}}$ denotes the element of F given by $r_T = 0$ for $T \neq S$ and $r_S = 1$. If $m \in \mathbb{Z}$, then let $F^{(m)} := \operatorname{span}_{\mathbb{C}}\{v_S : S \in \mathscr{S}_m\}$. We then have $F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}$, compare with [5, p. 310].

4.7. We now recall the definition of *wedging* and *contracting* operators. Our approach here is only notationally different from that of [5, p. 311]. On the other hand, we find our approach useful for relating these operators to our combinatorial construction on partitions, see §5 and especially Proposition 5.2.

To begin with, if $S = (s_1, s_2, \dots)$ is an ordered strictly decreasing sequence of integers and $j \in \mathbb{Z}$, then we use the notations $S \cup \{j\}$ and $S \setminus \{j\}$ to denote the ordered strictly decreasing sequence of integers determined by the sets $\{s_1, s_2, \dots\} \cup \{j\}$ and $\{s_1, s_2, \dots\} \setminus \{j\}$ respectively.

Next, given $j \in \mathbb{Z}$, define elements $f_j, f_j^* \in \mathrm{End}_{\mathbb{C}}(F)$, for $j \in \mathbb{Z}$, by:

$$f_j(v_S) := \begin{cases} (-1)^{\mathrm{count}(j,S)} v_{S\cup\{j\}} & \text{if } j \notin S \\ 0 & \text{if } j \in S \end{cases} \tag{4.7}$$

and

$$f_j^*(v_S) := \begin{cases} (-1)^{\mathrm{count}(j,S)} v_{S\setminus\{j\}} & \text{if } j \in S \\ 0 & \text{if } j \notin S, \end{cases} \tag{4.8}$$

and extending $\mathbb{C}$-linearly, compare with [5, §14.17], [1, p. 12], and [1, §A]. These endomorphisms have the properties that

$$f_i f_j^* + f_j^* f_i = \delta_{ij},\ f_i f_j + f_j f_i = 0,\ f_i^* f_j^* + f_j^* f_i^* = 0 \tag{4.9}$$

and

$$[f_i f_j^*, f_\ell f_k^*] = \delta_{j\ell} f_i f_k^* - \delta_{ik} f_\ell f_j^*, \tag{4.10}$$

for all $i, j, k, \ell \in \mathbb{Z}$, see [5, p. 311] for example.

For completeness, we note that (4.9) follows immediately from the definitions given in (4.7) and (4.8). On the other hand, (4.10) is a consequence of (4.9). Indeed, first note:

$$[f_i f_j^*, f_\ell f_k^*] = f_i f_j^* f_\ell f_k^* - f_i f_\ell f_k^* f_j^* + f_i f_\ell f_k^* f_j^* - f_\ell f_k^* f_i f_j^*$$

which can be rewritten using the second and third properties of (4.9) as:

$$f_i(f_j^* f_\ell + f_\ell f_j^*) f_k^* - f_\ell (f_i f_k^* + f_k^* f_i) f_j^*. \tag{4.11}$$

Applying the first property given in (4.9) to (4.11) yields the righthand side of (4.10).

Note also that the operators f_i, for $i \in \mathbb{Z}$, map $F^{(m)}$ to $F^{(m+1)}$, the operators f_i^*, for $i \in \mathbb{Z}$, map $F^{(m)}$ to $F^{(m-1)}$ whereas the operators $f_i f_j^*$, for $i, j \in \mathbb{Z}$, map $F^{(m)}$ to $F^{(m)}$.

4.8. The *infinite wedge representation* is the Lie algebra homomorphism

$$\rho : \widehat{\mathbf{gl}}(\infty) \to \mathrm{End}_{\mathbb{C}}(F)$$

determined by the conditions that

$$\rho((0, \mathrm{E}_{ij})) = \begin{cases} f_i f_j^* & \text{if } i \neq j \text{ or } i = j > 0 \\ f_i f_i^* - \mathrm{id}_F & \text{if } j = i \leqslant 0 \end{cases} \tag{4.12}$$

and

$$\rho((a,0)) = a\,\mathrm{id}_F, \tag{4.13}$$

for $a \in \mathbb{C}$, compare with [5, p. 313] for instance.

The fact that the above conditions (4.12) and (4.13) determine a representation of Lie algebras is deduced easily from property (4.10) above together with the definition of the 2-cocycle $\mathrm{c}(\cdot,\cdot)$, given in (2.6), and the fact that every element of $\mathbf{gl}(\infty)$ can be written as a finite linear combination of matrices of the form $\sum_{i\in\mathbb{Z}} a_i \mathrm{E}_{i,i+k}$, where $k \in \mathbb{Z}$ and $a_i \in \mathbb{C}$.

4.9. In what follows we refer to the restriction of ρ to the image of the morphism (2.7) as the *infinite wedge representation of the oscillator algebra* $\mathfrak{s}$.

5. Combinatorial Properties of the Operators $f_i f_j^*$

In this section we define and study certain operators on partitions. This construction will be used in our definition of the bosonic representation of the Lie algebra $\widehat{\mathbf{gl}}(\infty)$, see §6. Our main result is Proposition 5.2 which describes the combinatorics encoded in the vector

$$f_i f_j^*(v_S) = (-1)^\alpha v_T; \tag{5.1}$$

here

$$S := (s_1, s_2, \dots) \in \mathscr{S},$$

$i, j \in \mathbb{Z}$, are such that

$$j \in S \text{ and } i \notin S \setminus \{j\},$$

$$T := (S \setminus \{j\}) \cup \{i\},$$

and

$$\alpha := \operatorname{count}(i, S\setminus\{j\}) - \operatorname{count}(j, S).$$

As it turns out the combinatorics encoded in (5.1) are related to a certain skew diagram associated to the partition determined by S, see Proposition 5.1 and Proposition 5.2.

5.1. Let $m, i \in \mathbb{Z}$, let $\mathscr{P}_{m,i}$ denote the set

$$\mathscr{P}_{m,i} := \{\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P} : \lambda_k \neq i - m + k - 1 \text{ for all } k\},$$

and let $\mathscr{P}^*_{m,i}$ denote the set

$$\mathscr{P}^*_{m,i} := \{\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P} : \lambda_k = i - m + k - 1 \text{ for some } k\}.$$

Given $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P}$, define

$$\operatorname{count}_m(i, \lambda) := \#\{k : \lambda_k > i - m + k - 1\}. \tag{5.2}$$

The main idea behind (5.2) is that if $\lambda = \lambda(S)$ is the partition corresponding to a charge m semi-infinite monomial $S \in \mathscr{S}_m$, then

$$\operatorname{count}_m(i, \lambda) = \operatorname{count}(i, S), \tag{5.3}$$

where $\operatorname{count}(i, S)$ denotes the number of elements of S which are strictly greater than i, see (4.6). That (5.3) holds true is easy to check using (4.6) and Proposition 4.1 (b).

5.2. We now use (5.2) to define certain combinatorial operators on partitions. Precisely, if $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P}_{m,i}$, then define $p_{m,i}(\lambda)$ to be the partition $\mu = (\mu_1, \mu_2, \dots)$ where:

$$\mu_j = \begin{cases} \lambda_j - 1 & \text{for } j \leqslant \operatorname{count}_m(i, \lambda) \\ i - m + \operatorname{count}_m(i, \lambda) - 1 & \text{for } j = \operatorname{count}_m(i, \lambda) + 1 \\ \lambda_{j-1} & \text{for } j > \operatorname{count}_m(i, \lambda) + 1. \end{cases} \tag{5.4}$$

On the other hand, if $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P}^*_{m,i}$, then define $p^*_{m,i}(\lambda)$ to be the partition $\mu = (\mu_1, \mu_2, \dots)$ where:

$$\mu_j = \begin{cases} \lambda_j + 1 & \text{for } j \leqslant \operatorname{count}_m(i, \lambda) \\ \lambda_{j+1} & \text{for } j > \operatorname{count}_m(i, \lambda). \end{cases} \tag{5.5}$$

5.3. The following proposition is used in the proof of Proposition 5.2 which relates the combinatorial operators defined in §5.2 to the operators $f_i f_j^*$ described in §4.7 and (5.1).

Proposition 5.1. *Fix $m, i, j \in \mathbb{Z}$, $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathscr{P}^*_{m,j}$, let*

$$\mu := p^*_{m,j}(\lambda),$$

assume that $\mu \in \mathscr{P}_{m-1,i}$ and let

$$\nu := p_{m-1,i}(\mu) = p_{m-1,i} p^*_{m,j}(\lambda).$$

The following assertions hold true:

(a) *if $i < j$, then $\nu \subseteq \lambda$, the skew diagram $\lambda \setminus \nu$ is a border strip,*

$$\#(\lambda \setminus \nu) = j - i,$$

and

$$\operatorname{height}(\lambda \setminus \nu) = \operatorname{count}_{m-1}(i, \mu) - \operatorname{count}_m(j, \lambda);$$

(b) *if $i > j$, then $\lambda \subseteq \nu$, the skew diagram $\nu \setminus \lambda$ is a border strip,*

$$\#(\nu \setminus \lambda) = i - j,$$

and

$$\operatorname{height}(\nu \setminus \lambda) = \operatorname{count}_m(j, \lambda) - \operatorname{count}_{m-1}(i, \mu).$$

(c) *if $i = j$, then $\nu = \lambda$ and the skew diagrams $\nu \setminus \lambda$ and $\lambda \setminus \nu$ are empty.*

PROOF. By assumption we have

$$\mu := p^*_{m,j}(\lambda) \tag{5.6}$$

and

$$\nu := p_{m-1,i}(\mu) = p_{m-1,i}p^*_{m,j}(\lambda) = (\nu_1, \nu_2, \dots); \tag{5.7}$$

set

$$\alpha := \operatorname{count}_m(j, \lambda) \tag{5.8}$$

and

$$\beta := \operatorname{count}_{m-1}(i, \mu). \tag{5.9}$$

For (a), we have $i < j$. As a consequence, using the definitions (5.4) and (5.5), we deduce that the partition $\nu = (\nu_1, \nu_2, \dots)$ has the form:

$$\nu_k = \begin{cases} \lambda_k & \text{for } 1 \leqslant k \leqslant \alpha \\ \lambda_{k+1} - 1 & \text{for } \alpha + 1 \leqslant k \leqslant \beta \\ i - m + \beta & \text{for } k = \beta + 1 \\ \lambda_k & \text{for } k \geqslant \beta + 2. \end{cases} \tag{5.10}$$

Considering (5.10), it is clear that $\nu \subseteq \lambda$, that $\theta := \lambda \setminus \nu$ is a border strip, and that the number of rows of θ equals

$$\#[\alpha + 1, \beta + 1] = \beta - \alpha + 1; \tag{5.11}$$

it follows from (5.11) that

$$\operatorname{height}(\theta) = \beta - \alpha. \tag{5.12}$$

Next if θ_k denotes the number of elements in the kth row of θ, then $\theta_k = 0$ for $k \leqslant \alpha$ and $k \geqslant \beta + 2$. We also have:

$$\theta_k = \lambda_k - \lambda_{k+1} + 1, \tag{5.13}$$

for $\alpha+1 \leqslant k \leqslant \beta$,

$$\theta_{\beta+1} = \lambda_{\beta+1} - i - \beta + m, \tag{5.14}$$

and

$$\lambda_{\alpha+1} = j + \alpha - m. \tag{5.15}$$

Thus, using (5.13), (5.14), and (5.15), we have:

$$\sum_{k=\alpha+1}^{\beta+1} \theta_k = j + \alpha - m - i - \beta + m + \#[\alpha+1, \beta] = j - i,$$

whence

$$\#\theta = j - i.$$

For (b), we have $i > j$. As a consequence, using the definitions (5.4) and (5.5), we deduce that the partition $\nu = (\nu_1, \nu_2, \dots)$ is defined by:

$$\begin{cases} \lambda_k & \text{for } 1 \leqslant k \leqslant \beta \\ i + \beta - m & \text{for } k = \beta + 1 \\ \lambda_{k-1} + 1 & \text{for } \beta + 1 < k \leqslant \alpha + 1 \\ \lambda_k & \text{for } k \geqslant \alpha + 2. \end{cases} \tag{5.16}$$

Considering (5.16), it is clear that $\lambda \subseteq \nu$, that $\theta := \nu \setminus \lambda$ is a border strip, and that the number of rows of θ equals

$$\#[\beta+1, \alpha+1] = \alpha - \beta + 1. \tag{5.17}$$

Thus

$$\operatorname{height}(\theta) = \alpha - \beta. \tag{5.18}$$

Next let θ_k denote the number of elements in the kth row of θ. Then $\theta_k = 0$ for $k \leqslant \beta$ and $k > \alpha + 1$. We also have:

$$\theta_{\beta+1} = i + \beta - m - \lambda_{\beta+1}, \tag{5.19}$$

$$\theta_k = \lambda_{k-1} + 1 - \lambda_k, \tag{5.20}$$

for $\beta + 1 < k \leqslant \alpha + 1$, and

$$\lambda_{\alpha+1} = j + \alpha - m. \tag{5.21}$$

Using (5.19), (5.20), and (5.21), it follows that

$$\sum_{k=\beta+1}^{\alpha+1} \theta_k = i + \beta - m - j - \alpha + m + \#[\beta+2, \alpha+1] = i - j$$

so that

$$\#\theta = i - j.$$

Assertion (c) is trivial. $\square$

5.4. Example. Recall, see §4.4, that

$$S = (4, 3, 1, 0, -2, -3, -5, -7, -8, \dots)$$

is the element of $\mathscr{S}_0$ corresponding to the partition

$$\lambda = (4, 4, 3, 3, 2, 2, 1) \in \mathscr{P}_{19},$$

whose Young diagram is pictured in §4.4. To compute $f_{-1} f_3^*(v_S)$ note that

$$T = (S \setminus \{3\}) \cup \{-1\} = (4, 1, 0, -1, -2, -3, -5, -7, -8, \dots),$$

$\operatorname{count}(3, S) = 1$ and $\operatorname{count}(-1, S \setminus \{3\}) = 3$. We conclude

$$f_{-1} f_3^*(v_S) = (-1)^{3-1} v_T = v_T. \tag{5.22}$$

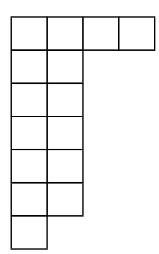
To see the combinatorics encoded in (5.22) first note that

$$\nu := \lambda(T),$$

the partition corresponding to T, then

$$\nu = (4, 2, 2, 2, 2, 2, 1)$$

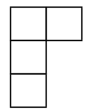
which has Young diagram



and $\nu \subseteq \lambda$. The skew diagram $\theta := \lambda \setminus \nu$ is the set

$$\{\{2, 3\}, \{2, 4\}, \{3, 3\}, \{4, 3\}\}$$

which can be represented pictorially as:



Note that the skew diagram θ is a border strip and $\text{height}(\theta) = 2$. If we now identify S with λ and T with ν, then (5.22) takes the form

$$f_{-1} f_3^*(v_\lambda) = (-1)^{\text{height}(\theta)} v_\nu. \tag{5.23}$$

Suppose now that we wish to compute $f_{-1} f_{-3}^*(v_S)$. In this case, $\text{count}(-3, S) = 5$, $\text{count}(-1, S \setminus \{-3\}) = 4$ and hence

$$f_{-1} f_{-3}^*(v_S) = -1 v_T, \tag{5.24}$$

where

$$T = (4, 3, 1, 0, -1, -2, -5, -7, -8, \dots).$$

The combinatorics encoded in (5.24) is similar to that encoded in (5.22), but there is one difference which amounts to the fact that $-1 > -3$ while $3 > -1$. In more detail, if

$$\nu := \lambda(T),$$

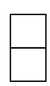
then

$$\nu = (4, 4, 3, 3, 3, 3, 1),$$

$\lambda \subseteq \nu$, and the skew diagram $\theta := \nu \setminus \lambda$ is

$$\theta = \{\{5, 3\}, \{6, 3\}\}$$

which is a border strip. The border strip θ can be pictured pictorially as:



and has height equal to one. If we identify S with λ and T with ν, then (5.24) takes the form

$$f_{-1} f_{-3}^*(v_\lambda) = (-1)^{\text{height}(\theta)} v_\nu.$$

5.5. Example 5.4 generalizes:

Proposition 5.2. *Suppose that $S = (s_1, s_2, \dots) \in \mathscr{S}$ and $i, j \in \mathbb{Z}$. Then $f_i f_j^*(v_S) \neq 0$ if and only if $j \in S$, and $i \notin S \setminus \{j\}$. In addition assume that $f_i f_j^*(v_S) \neq 0$, let $T := (S \setminus \{j\}) \cup \{i\}$, let λ and ν be the partitions determined by S and T respectively, and denote v_S by v_λ and v_T by v_ν. The following assertions hold true:*

(a) *If $i < j$, then $\nu \subseteq \lambda$, the skew diagram $\lambda \setminus \nu$ is a border strip of length $j - i$ and*

$$f_i f_j^*(v_\lambda) = (-1)^{\text{height}(\lambda\setminus\nu)} v_\nu;$$

(b) *If $j < i$, then $\lambda \subseteq \nu$, the skew diagram $\nu \setminus \lambda$ is a border strip of length $i - j$ and*

$$f_i f_j^*(v_\lambda) = (-1)^{\text{height}(\nu\setminus\lambda)} v_\nu.$$

PROOF. The proposition is a consequence of Proposition 4.1, the discussion given in §5.1 and Proposition 5.1. In particular, using Proposition 4.1 (b) in conjunction with (5.10) and (5.16), depending on whether $i < j$ or $j < i$, we compute that

$$\nu = p_{i,m-1} p_{j,m}^*(\lambda).$$

The conclusion of Proposition 5.2 then follows from Proposition 5.1, (5.3) and (5.1). □

6. **The Bosonic Representation of $\widehat{\mathfrak{gl}}(\infty)$** We now provide an application of our combinatorial construction given in §5. Indeed, we use this construction to prove the boson-fermion correspondence which we state as Theorem 6.2.

6.1. To begin with, let $A := \mathbb{C}[z, z^{-1}]$ and

$$B := A \otimes_{\mathbb{C}} \Lambda = \mathbb{C}[z, z^{-1}, h_1, h_2, \ldots].$$

The *bosonic representation of the oscillator algebra* is the Lie algebra homomorphism

$$\xi_0 : \mathfrak{s} \to \operatorname{End}_{\mathbb{C}}(B) \tag{6.1}$$

determined by:

$$\xi_0((0, t^k)) = p_k^\perp = k\frac{\partial}{\partial p_k}, \text{ for } k > 0;$$

$$\xi_0((0, t^k)) = p_{-k}, \text{ for } k < 0;$$

$$\xi_0((0, 1)) = z\frac{\partial}{\partial z};$$

and

$$\xi_0((1, 0)) = 1,$$

compare with [5, p. 314] or [6, Lecture 5, p. 46].

6.2. The first step to proving Theorem 6.2 is to define operators

$$b_i \in \operatorname{End}_{\mathbb{C}}(B)$$

by the rule:

$$b_i(z^m s_\lambda) = \begin{cases} (-1)^{\mathrm{count}_m(i,\lambda)} z^{m+1} s_{p_{m,i}(\lambda)} & \text{for } \lambda \in \mathscr{P}_{m,i} \\ 0 & \text{for } \lambda \notin \mathscr{P}_{m,i}. \end{cases} \tag{6.2}$$

Similarly define operators

$$b_i^* \in \mathrm{End}_{\mathbb{C}}(B)$$

by the rule

$$b_i^*(z^m s_\lambda) \begin{cases} (-1)^{\mathrm{count}_m(i,\lambda)} z^{m-1} s_{p^*_{m,i}(\lambda)} & \text{for } \lambda \in \mathscr{P}^*_{m,i} \\ 0 & \text{for } \lambda \in \mathscr{P}^*_{m,i}. \end{cases} \tag{6.3}$$

As in (4.9) and (4.10), we have the relations

$$b_i b_j^* + b_j^* b_i = \delta_{ij},\ b_i b_j + b_j b_i = 0,\ b_i^* b_j^* + b_j^* b_i^* = 0, \tag{6.4}$$

and

$$[b_i b_j^*, b_\ell b_k^*] = \delta_{j\ell} b_i b_k^* - \delta_{ik} b_\ell b_j^*, \tag{6.5}$$

for all $i, j, k, \ell \in \mathbb{Z}$. Indeed, as in (4.9), (6.4) follows immediately from the definitions while (6.5) is deduced from (6.4).

6.3. Example. As in §4.4, if $\lambda = (4, 4, 3, 3, 2, 2, 1)$, then

$$\mathrm{count}_0(3, \lambda) = 1$$

and

$$b_3^*(s_\lambda) = -z^{-1} s_\mu,$$

where μ is the partition $\mu = (5, 3, 3, 2, 2, 1)$. Also, $\mathrm{count}_{-1}(-1, \mu) = 3$,

$$b_{-1}(z^{-1} s_\mu) = -s_\nu,$$

where $\nu = (4, 2, 2, 2, 2, 1)$, and

$$b_{-1} b_3^*(s_\lambda) = s_\nu.$$

6.4. The key point in the proof of Theorem 6.2 is the following observation which is a consequence of Proposition 5.2. The point is that if $S \in \mathscr{S}$, k a nonzero integer and $\mathfrak{s}_k := (0, t^k) \in \mathfrak{s}$, then

$$\delta_0(\mathfrak{s}_k)(v_S) = \sum_{\text{finite}} f_\ell f^*_{\ell+k}(v_S) \tag{6.6}$$

and we now give a combinatorial description of this finite set:

PROPOSITION 6.1. *Suppose that $S \in \mathscr{S}_m$. The following assertions hold true:*

(a) *If $k > 0$, then*

$$\mathfrak{s}_k(v_S) = \sum_{\text{finite}} (-1)^{\operatorname{height}(\lambda(S)\setminus\lambda(T))} v_T,$$

where the finite sum is taken over all $T \in \mathscr{S}_m$, which have the property that $\lambda(T) \subseteq \lambda(S)$, and $\lambda(S) \setminus \lambda(T)$ is a border strip of length k.

(b) *If $k < 0$, then*

$$\mathfrak{s}_k(v_S) = \sum_{\text{finite}} (-1)^{\operatorname{height}(\lambda(T)\setminus\lambda(S))} v_T$$

where the finite sum is taken over all $T \in \mathscr{S}_m$ with the property that $\lambda(S) \subseteq \lambda(T)$ and $\lambda(T) \setminus \lambda(S)$ is a border strip of length $|k|$.

PROOF. To begin with, note that for both (a) and (b), Proposition 5.2 implies that each summand of (6.6) contributes a summand of the desired form.

To establish Proposition 6.1 it thus remains to show that, conversely, each border strip of the shape asserted in the proposition appears as a summand of (6.6).

To this end, consider the case that $k > 0$. Let λ be the partition corresponding to S, suppose that $\nu \subseteq \lambda$ is such that $\theta := \lambda \setminus \nu$ is a border strip of length k. Let $\theta_{n'}$ denote the number of elements in the n'th row of θ. Let $n := \min\{j : \nu_j \neq \lambda_j\}$. Then $\theta_{n'} = 0$ for $n' < n$ and $\theta_n \neq 0$; set

$$\ell := \theta_n - k - n + m + \lambda_{n+1}. \tag{6.7}$$

We then compute, using the definitions (5.4) and (5.5) together with the fact that θ is a border strip, that

$$\nu = p_{m-1,\ell} p^*_{m,\ell+k}(\lambda); \tag{6.8}$$

compare with (5.13) and (5.14).

Thus if T is the element of $\mathscr{S}_m$ corresponding to ν, then

$$(-1)^{\operatorname{height}(\lambda\setminus\nu)} v_T = f_\ell f^*_{\ell+k}(v_S), \tag{6.9}$$

by Proposition 5.2 (a).

Next suppose that $k < 0$. Again let λ be the partition corresponding to S, suppose that $\nu \supseteq \lambda$ is such that $\theta := \nu \setminus \lambda$ is a border strip of length $|k|$, let T be the element of $\mathscr{S}_m$ corresponding to ν and let $n := \min\{j : \nu_j \neq \lambda_j\}$. Let $\theta_{n'}$ denote the number of elements in the n'th row of θ and set

$$\ell := \theta_n - (n-1) + \lambda_n + m. \tag{6.10}$$

We then compute, using the definitions (5.4) and (5.5) together with the fact that θ is a border strip, that:

$$\nu = p_{m-1,\ell} p^*_{m,\ell+k}(\lambda); \tag{6.11}$$

compare with (5.19) and (5.20).

Thus if T is the element of $\mathscr{S}_m$ corresponding to ν, then

$$(-1)^{\mathrm{height}(\nu\setminus\lambda)} v_T = f_\ell f^*_{\ell+k}(v_S), \tag{6.12}$$

by Proposition 5.2 (b). $\square$

6.5. Using the theory we have developed thus far we can prove the boson-fermion correspondence.

THEOREM 6.2. *The bosonic representation*

$$\xi_0 : \mathfrak{s} \to \mathrm{End}_{\mathbb{C}}(B),$$

namely (6.1)*, of the oscillator algebra extends to a representation*

$$\xi : \widehat{\mathbf{gl}}(\infty) \to \mathrm{End}_{\mathbb{C}}(B)$$

of the Lie algebra $\widehat{\mathbf{gl}}(\infty)$*. More precisely, the Lie algebra* $\widehat{\mathbf{gl}}(\infty)$ *admits a representation* $\xi : \widehat{\mathbf{gl}}(\infty) \to \mathrm{End}_{\mathbb{C}}(B)$ *with the property that the diagram*

$$\begin{tikzcd}
\mathfrak{s} \arrow[d, "\delta_0"'] \arrow[dr, "\xi_0"] & \\
\widehat{\mathbf{gl}}(\infty) \arrow[r, "\xi"] & \mathrm{End}_{\mathbb{C}}(B)
\end{tikzcd}$$

commutes. In addition, the $\mathbb{C}$*-linear isomorphism*

$$\sigma : F \to B$$

defined by

$$v_S \mapsto z^m s_{\lambda(S)},$$

for $m = \mathrm{charge}(S)$ *and* $\lambda(S)$ *the partition determined by the semi-infinite monomial* S*, is an isomorphism of* $\widehat{\mathbf{gl}}(\infty)$*-modules.*

PROOF. Consider the representation

$$\xi : \widehat{\mathbf{gl}}(\infty) \to \mathrm{End}_{\mathbb{C}}(B)$$

determined by the conditions that:

$$\xi((0, \mathrm{E}_{ij})) = \begin{cases} b_i b_j^* & \text{if } i \neq j \text{ or } i = j > 0 \\ b_i b_i^* - \mathrm{id}_B & \text{if } j = i \leqslant 0 \end{cases}$$

and

$$\xi((a,0)) = a\,\mathrm{id}_B,$$

for $a \in \mathbb{C}$. The fact that ξ is a representation follows from the relations given in (6.4). The fact that ξ extends the representation ξ_0 follows from Proposition 6.1 and the Murnaghan-Nakayama rule (2.1) and (2.2).

For the second assertion, fix $i, j \in \mathbb{Z}$ and assume that $f_i f_j^*(v_S) \neq 0$. We then have that

$$f_i f_j^*(v_S) = (-1)^\alpha v_T,$$

where

$$T := (S \setminus \{j\}) \cup \{i\},$$

and

$$\alpha := \mathrm{count}(i, S \setminus \{j\}) - \mathrm{count}(j, S);$$

let $\lambda = \lambda(S)$ be the partition corresponding to S and let $\nu = \lambda(T)$ be the partition corresponding to T.

In this setting, the operator $p^*_{m,j}$ is defined on the partition λ and the operator $p_{m-1,i}$ is defined on the partition $p^*_{m,j}(\lambda)$. In addition

$$\nu = p_{m-1,i} p^*_{m,j}(\lambda).$$

On the other hand we have that

$$\sigma(v_S) = z^m s_\lambda.$$

Considering the definitions of the operators b_i and b_j^*, we then deduce that

$$b_i b_j^*(z^m s_\lambda) = (-1)^\alpha z^m s_\nu$$

which is what we wanted to show. □

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Department of Mathematics and Statistics, University of New Brunswick, Fredericton, NB, Canada E3B 5A3
e-mail: n.grieve@unb.ca