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Cartan-Remez Type Inequalities for Analytic and Plurisubharmonic Functions

Alexander Brudnyi

Presented by Pierre Milman, FRSC

Abstract. Recently there has been a considerable interest in Cartan-Remez type inequalities in connection with various problems of analysis. In this paper we formulate and prove several basic results in this area and describe some of their applications. The text is based on the material of the minicourse given by the author at the workshop on Analytic Microlocal Analysis held at the Northwestern University in May 2013.

Résumé. Récemment, il y a eu un intérêt considérable aux inégalités de types de Cartan-Remez dans le cadre de divers problèmes de l'analyse. Dans cet article, nous formulons et nous démontrons plusieurs résultats de base dans ce domaine et nous décrivons certaines de leurs applications. Le texte est basé sur le matériau de la mini-course donnée par l'auteur à l'atelier sur l'analyse analytique microlocale tenu à l'Université Northwestern en mai 2013.

1. **Introduction** This paper is devoted to polynomial type inequalities for analytic and plurisubharmonic functions. The term is referred to Markov, Bernstein, Cartan and Remez type inequalities and their numerous generalizations. Historically, the classical univariate inequalities for polynomials have appeared in approximation theory and for a long time have been considered as technical tools for proofs of Bernstein type inverse theorems. At the present time, polynomial type inequalities have found a lot of important applications in areas which are well apart from approximation theory such as convex geometry (the famous slice problem), algebraic geometry (characterization of algebraic varieties, Bezout type theorems), function spaces (Sobolev type embeddings and trace theorems), potential theory (Bernstein-Walsh type inequalities, capacity estimates), transcendental number theory, differential equations (subelliptic estimates, the second part of Hilbert's sixteenth problem), to name but a few (see e.g., the Introduction to [6] and references therein, results and references in [5, Ch. 2, 9, 10], [1, 3, 4, 16], etc.).

In this expository paper we mostly focus on Cartan-Remez type inequalities presenting some basic results in the area together with some applications.

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Specifically, in the next section, we formulate and prove a generalization of the classical Cartan lemma which allows to obtain effective lower bounds for logarithmic potentials and subharmonic functions. Section 3 deals with the classical Remez polynomial inequality and its generalizations for certain classes of plurisubharmonic functions. Finally, in Section 4 we describe several applications of Cartan-Remez type inequalities.

2. Cartan Lemma

The classical Cartan Lemma [13] estimates the size of a polynomial lemniscate in terms of one dimensional Hausdorff content. This result can be extended to estimate 'massivity' of sublevel sets of subharmonic functions (see, e.g., [24]). In this section, we present a general approach allowing to obtain such estimates based on a Vitali-type covering lemma (see Lemma 2.1 below) discovered by E. A. Gorin in 1982 (see, e.g., [19]).

2.1. Vitali-type covering lemma Let (X, d) be a metric space. By

$$\bar{B}_r(x) := \{y \in X \, : \, d(y,x) \le r, \, r \ge 0\} \subset X$$

we denote the closed ball with center x and radius r. Let μ be a Borel measure on X with $\mu(X) = A < \infty$. Consider a continuous strictly increasing nonnegative function φ on $[0,\infty)$, $\varphi(0) = 0$, $\lim_{t\to\infty} \varphi(t) > A$, referred to as a *majorant.* For each point $x \in X$ we set

$$\tau(x) := \sup\{t \in [0,\infty) \, : \, \mu(\bar{B}_t(x)) \ge \varphi(t)\}. \tag{2.1}$$

It is easy to see that $\mu(\bar{B}_{\tau(x)}(x)) = \varphi(\tau(x))$ and $\sup_{x\in X} \tau(x) \le \varphi^{-1}(A)$.

A point $x \in X$ is said to be *regular* (with respect to μ, φ) if $\tau(x) = 0$, i.e. $\mu(\bar{B}_t(x)) < \varphi(t)$ for all $t > 0$.

The next result shows that the set of regular points is sufficiently large.

LEMMA 2.1 (E. A. Gorin). *Suppose that not all points of X are regular. Then given $\gamma \in (0, 1/2)$ there is a sequence of balls $B_k = \bar{B}_{t_k}(x_k)$, $k = 1, 2, \dots,$ which collectively cover all irregular points of X such that $\sum_{k\ge1} \varphi(\gamma t_k) < A$.*

PROOF. Let $0 < \alpha < 1$, $\beta > 2$ be such that $\gamma < \alpha/\beta$. We set $B_0 = \varnothing$ and assume that the balls $B_0, \dots, B_{k-1}$ have been constructed. If either $X = \cup_{0\le j\le k-1} B_j$ or $\tau_k := \sup\{\tau(x) \, : \, x \notin \cup_{0\le j\le k-1} B_j\} = 0$, then $k - 1 \ge 1$ and $B_1, \dots, B_{k-1}$ is the required sequence. For otherwise, there exists a point $x_k \notin \cup_{0\le j\le k-1} B_j$ such that $\tau(x_k) > \alpha\tau_k$. We set $t_k = \beta\tau_k$ and $B_k = \bar{B}_{t_k}(x_k)$. Clearly, $\{\tau_k\}$ is a nonincreasing sequence. Also, the family $\{B_{\tau_k}(x_k)\}$ consists of pairwise disjoint balls. Indeed, if $l > k$, then $x_l \notin B_k$, i.e., $d(x_l, x_k) \ge \beta\tau_k > 2\tau_k \ge \tau_k + \tau_l$.

Thus, $\bar{B}_{\tau_k}(x_k) \cap \bar{B}_{\tau_l}(x_l) = \varnothing$ by the triangle inequality for d. Now,

$$\begin{aligned}\sum_{k\geq 1}\varphi(\gamma t_k) &< \sum_{k\geq 1}\varphi((\alpha/\beta)t_k) < \sum_{k\geq 1}\varphi(\tau(x_k)) \leq \sum_{k\geq 1}\mu(\bar{B}_{\tau_k}(x_k)) \\ &= \mu\left(\bigcup_{k\geq 1}\bar{B}_{\tau_k}(x_k)\right) \leq A;\end{aligned}$$

consequently, if $\{B_k\}$ is infinite, then $\lim_{k\to\infty}\tau_k = 0$, i.e., for each point $x \notin \cup_{k\geq 1}B_k$, $\tau(x) = 0$, that is, x is a regular point. □

Remark 2.2. If $\operatorname{supp}\mu \subset \{x_1, \ldots, x_n\} \subset X$, then as follows from the proof, $\#\{B_k\} \leq n$ and $\operatorname{supp}\mu \subset \cup_k B_k$.

2.2. Lower bounds for logarithmic potentials Let μ be a Borel measure on X with $\mu(X) = k < \infty$ such that

$$\int_X \ln^+ d(x,\zeta)d\mu(\zeta) < \infty \quad \text{for all} \quad x \in X;$$

here $\ln^+ t = \max\{0, \ln t\}$, $t \geq 0$. Consider the logarithmic potential

$$u(x) = \int_X \ln d(x,\zeta)d\mu(\zeta), \quad x \in X.$$

(For each $x \in X$ the above integral exists but may be equal to $-\infty$.)

Using Lemma 2.1 one proves the following result.

Theorem 2.3. *Suppose the majorant $\varphi : [0,\infty) \to [0,\infty)$ satisfies*

$$I_\varphi := \int_0^1 \frac{\varphi(t)}{t}\,dt < \infty. \tag{2.2}$$

Fix $\gamma \in (0, 1/2)$. Given $H > 0$ there is a family of closed balls B_j with radii r_j satisfying

$$\sum_{j\geq 1}\varphi\left(\frac{\gamma r_j}{H}\right) < I_\varphi \tag{2.3}$$

such that

$$u(x) \geq k\ln\left(\frac{H}{e}\right) \quad \textit{for all} \quad x \in X \setminus \bigcup_j B_j. \tag{2.4}$$

Remark 2.4. One easily shows that condition (2.2) implies

$$\lim_{t\to 0^+}\varphi(t)\ln t = 0. \tag{2.5}$$

PROOF. Using the majorant

$$\varphi_H(t) := \frac{k \cdot \varphi\left(\frac{t}{H}\right)}{I_\varphi}, \qquad t \geq 0,$$

we cover all irregular points with respect to φ_H and μ of X by closed balls B_j with radii r_j, $j \in \mathbb{N}$, according to Lemma 2.1 and prove that the required inequality is valid for any regular point x. Since by the lemma

$$\sum_{j\geq 1} \varphi\left(\frac{\gamma r_j}{H}\right) = \frac{I_\varphi}{k} \cdot \sum_{j\geq 1} \varphi_H(\gamma r_j) < I_\varphi,$$

this will complete the proof.

For a regular point x we set $n(t;x) := \mu(\bar{B}_t(x))$. Then for each $N \geq \max\{1, H\}$ we have

$$u(x) \geq \int_{\bar{B}_N(x)} \ln d(x,\zeta) d\mu(\zeta) = \int_0^N \ln t \, dn(t;x) = n(t;x) \ln t\big|_0^N - \int_0^N \frac{n(t;x)}{t}\, dt.$$

Since $n(t;x) < \varphi_H(t)$, condition (2.5) and the latter imply

$$u(x) \geq n(N;x) \ln N - \int_0^N \frac{n(t;x)}{t}\, dt.$$

In addition, $n(t;x) \leq n(N;x)$ for $t \leq N$. Therefore,

$$u(x) \geq n(N;x) \ln N - \int_0^H \frac{\varphi_H(t)}{t}\, dt - \int_H^N \frac{n(N;x)}{t}\, dt = n(N;x) \ln N - k$$
$$-n(N;x) \ln N + n(N;x) \ln H = -k + n(N;x) \ln H.$$

Letting here $N \to \infty$ and taking into account that $\lim_{N\to\infty} n(N;x) = k$, we obtain the required result. □

Let us consider several examples.

EXAMPLE 2.5. (A) Let $p(z) = \prod_{j=1}^{k}(z - c_j)$, $z \in \mathbb{C}$, be a monic holomorphic polynomial of degree k. Then from Theorem 2.3 for $d(z_1, z_2) := |z_1 - z_2|$, $z_1, z_2 \in \mathbb{C}$, $\mu = \sum_{j=1}^{k} \delta_{c_j}$ and $\varphi(t) = t^s$, $0 < s \leq 1$, one obtains:

Given $H > 0$, $s > 0$ there is a family of closed disks $\{D_j\}_{j=1}^m$, $m \leq k$, with radii r_j satisfying $\sum r_j^s < \frac{(2H)^s}{s}$ such that

$$|p(z)| \geq \left(\frac{H}{e}\right)^k \quad \textit{for all} \quad z \in \mathbb{C} \setminus \bigcup_j D_j.$$

For $s = 1$ this statement coincides with the classical *Cartan Lemma*.

(B) Let $B(z) = \prod_{j=1}^{k} \frac{z-c_j}{1-\bar{c}_j z}$, all $c_j, z \in \mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$, be a finite Blaschke product. Applying Theorem 2.3 to the pseudohyperbolic distance $d(z_1, z_2) := \left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right|$ on $\mathbb{D}$ and $\mu = \sum_{j=1}^{k} \delta_{c_j}$, and $\varphi(t) = t^s$, $0 < s \leq 1$, one obtains:

Given $H > 0$, $s > 0$ there is a family of closed disks $\{D_j\}_{j=1}^{m}$, $m \leq k$, in $\mathbb{D}$ with radii r_j satisfying $\sum r_j^s \leq \dfrac{(2H)^s}{s}$ such that

$$|B(z)| \geq \left(\frac{H}{e}\right)^k \quad \textit{for all} \quad z \in \mathbb{D} \setminus \bigcup_j D_j.$$

2.3. Cartan-type estimates for subharmonic functions Theorem 2.3 can be applied to prove the following result.

Let $u \not\equiv 0$ be a logarithmically subharmonic function on the disk $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$ (i.e., $\ln u \not\equiv -\infty$ is subharmonic on $\mathbb{D}_R$). It is well known that the Laplacian of $\ln u$ (determined in the distributional sense) is represented by a (unique) Borel measure $\mu_{\ln u}$ on $\mathbb{D}_R$, called the *Riesz measure* of $\ln u$, see, e.g., [21]. Fix positive α, β such that $\alpha < \beta < 1$. For $\lambda \in (0,1)$ we set $M_u(\lambda R) = \sup_{x \in \mathbb{D}_{\lambda R}} u(x)$.

Theorem 2.6. *Let $H \leq \beta e$ and s be positive numbers. Fix $\gamma \in (0, 1/2)$. There is a family of closed disks $\{D_j\}$ with $\sum r_j^s < \dfrac{\left((H/\gamma)R\right)^s}{s}$, where r_j is the radius of D_j, such that*

$$u(z) \geq M_u(\beta R) \left(\frac{M_u(\alpha R)}{M_u(\beta R)}\right)^{\left(\frac{\beta+\alpha}{\beta-\alpha}\right)^2} \cdot \left(\frac{H}{\beta e}\right)^{\mu_{\ln u}(\mathbb{D}_{\beta R})} \tag{2.6}$$

for all $z \in \mathbb{D}_{\alpha R} \setminus \bigcup_j D_j$.

Proof. It suffices to prove the result for $v(z) = u(\beta R z)$, $z \in \mathbb{D}_\delta$, and the disks $\mathbb{D}_\lambda \subset \mathbb{D} \subset \mathbb{D}_\delta$, where $\lambda := \frac{\alpha}{\beta}$, $\delta := \frac{1}{\beta}$. Due to the *Riesz representation theorem* for $\ln v$, see, e.g., [21],

$$\ln v(z) = \int_{\mathbb{D}} \ln \left|\frac{z-\zeta}{1-\bar{z}\zeta}\right| d\mu_{\ln v}(\zeta) + h(z), \quad z \in \mathbb{D}, \tag{2.7}$$

where h is a bounded harmonic function on $\mathbb{D}$ whose boundary values coincide a.e. with $\ln u$. By Theorem 2.3 the first term on the right-hand side of (2.7) is bounded from below by $\mu_{\ln v}(\mathbb{D}) \ln\left(\frac{H}{\beta e}\right)$ apart from the union of closed pseudo-hyperbolic balls $B_j \subset \mathbb{D}$ of radii r_j satisfying $\sum r_j^s < \frac{(H/(\beta\gamma))^s}{s}$. Since each B_j is the subset of the closed Euclidean disk $\tilde{D}_j$ centered at the same point and of the same radius as B_j, this inequality is also valid for all $z \in \mathbb{D} \setminus \bigcup_j \tilde{D}_j$. We set $D_j := \beta R \cdot \tilde{D}_j$.

Next, by our definition, $M_u(\beta R) = M_h(1)$ and $M_u(\alpha R) = M_v(\gamma)$. These and (2.7) imply that $M_h(\gamma) \geq M_u(\alpha R)$ and the function $g := -h + \ln M_u(\beta R)$ is nonnegative harmonic on $\mathbb{D}$. Applying to g the classical Harnack inequality, we estimate e^h from below on $\mathbb{D}_\gamma$ by the product of the first two terms on the right-hand side of (2.6). Combining this with the above estimate of the first term on the right-hand side of (2.7), taking into account that $\mu_{\ln v}(\mathbb{D}) = \mu_{\ln u}(\mathbb{D}_{\beta R})$ and going back from v to u, we obtain (2.6), see, e.g., [6, Th. 3.1] for details. □

Remark 2.7. If $u = |f|$, where $f \not\equiv 0$ is holomorphic on $\mathbb{D}_R$, then $\mu_{\ln u}(\mathbb{D}_{\beta R})$ is the number of zeros (counting with multiplicities) of f in $\mathbb{D}_{\beta R}$. In general, using methods of potential theory, see, e.g., [18], one obtains

$$\mu_{\ln u}(\mathbb{D}_{\beta R}) \leq \frac{2\pi}{\ln(1/\beta)} \cdot (\ln M_u(R) - \ln M_u(\beta R)). \tag{2.8}$$

An analog of Theorem 2.6 for multivariate plurisubharmonic functions was proved by A. Zeriahi [29].

3. Remez-type Inequality

3.1. Inequality for polynomials Let

$$T_k(x) = \frac{(x - \sqrt{x^2-1})^k + (x + \sqrt{x^2-1})^k}{2}, \quad x \in \mathbb{R},$$

be the Chebyshev polynomial of degree k.

The following result was proved by E. Remez [26] for $n = 1$ and by Yu. Brudnyi and M. Ganzburg [11] for $n \geq 1$.

Theorem 3.1. *Let $V \subset \mathbb{R}^n$ be a compact convex body and $S \subset V$ be a subset of relative Lebesgue measure*

$$\ell := \frac{|S|_n}{|V|_n} > 0$$

(here $|\cdot|_n$ is the Lebesgue measure on $\mathbb{R}^n$).

For every polynomial p of degree k on $\mathbb{R}^n$ the sharp inequality

$$\max_V |p| \leq T_k\left(\frac{1 + \sqrt[n]{1-\ell}}{1 - \sqrt[n]{1-\ell}}\right) \max_S |p| \tag{3.1}$$

holds.

Proof for $n = 1$. In this case $V \subset \mathbb{R}$ is a compact interval. Let $\xi \in V$ be an extreme point for a polynomial p of degree k on $\mathbb{R}$, i.e., $|p(\xi)| = \max_V |p|$.

We consider two cases.

(a) ξ is one of the endpoints of V.

Making the change of variable, if necessary, without loss of generality we may assume that $V = [0,1]$ and $\xi = 0$. Then $S \subset [0,1]$ is of measure ℓ. Without loss of generality, we may also assume that S is closed. Consider the normalized Chebyshev polynomial $T_k^I(x) := T_k\big(\frac{2x}{l} - 1\big)$, $x \in \mathbb{R}$, of degree k for $I := [0,\ell]$, and let

$$0 =: x_1 < x_2 < \cdots < x_{k+1} := \ell$$

be points where T_k^I assumes alternatively the values ± 1. Then we choose points $\hat{x}_1 < \hat{x}_2 < \cdots < \hat{x}_{k+1}$ in S as follows. Set $\hat{x}_1 := \sup S \, (\in S)$ and determine $\hat{x}_i$ for $2 \le i \le k+1$ by the condition

$$\big|S \cap [\hat{x}_1, \hat{x}_i]\big|_1 = x_i - x_1.$$

These points are correctly defined, since $|S|_1 = \ell = x_{k+1} - x_1$. Due to this choice,

$$1 - x_j \ge \hat{x}_1 - x_j \ge \hat{x}_j \quad \text{and} \quad |\hat{x}_i - \hat{x}_j| \ge |x_i - x_j|. \tag{3.2}$$

Further, each $\hat{x}_i \in S$ and therefore $|p(\hat{x}_i)| \le \sup_S |p|$.

Using Lagrange interpolation we then have from (3.2)

$$|p(0)| \le \sum_{i=1}^{k+1} \bigg(\prod_{i \ne j} \frac{|\hat{x}_j|}{|\hat{x}_i - \hat{x}_j|} \bigg) |p(\hat{x}_j)| \le \Bigg(\sum_{i=1}^{k+1} \bigg(\prod_{j \ne i} \frac{1 - x_j}{|x_i - x_j|} \bigg) \big|T_k^I(x_i)\big| \Bigg) \sup_S |p|.$$

The sum in the right-hand side equals

$$\Bigg| \sum_{i=1}^{k+1} \bigg(\prod_{j \ne i} \frac{1 - x_j}{x_i - x_j} \bigg) T_k^I(x_i) \Bigg| = \big|T_k^I(1)\big| = T_k\Big(\frac{2-\ell}{\ell}\Big),$$

and the result follows.

(b) ξ is an interior point of V.

As before, we may and will assume that $V = [0,1]$. Then at least one of the fractions

$$\ell_1 := \frac{\big|[0,\xi] \cap S\big|}{\xi}, \quad \ell_2 := \frac{\big|[\xi,1] \cap S\big|}{1-\xi}$$

is greater than or equal to

$$\ell = \frac{\big|[0,\xi] \cap S\big| + \big|[\xi,1] \cap S\big|}{\xi + (1-\xi)}.$$

If, e.g., $\ell_1 \ge \ell$, then we apply the inequality proved in (a) to the interval $I_1 := [0,\xi]$ and the subset $S_1 := S \cap [0,\xi]$ to have

$$\max_{[0,1]} |p| = |p(\xi)| \le T_k\Big(\frac{2-\ell_1}{\ell_1}\Big) \max_{S_1} |p| \le T_k\Big(\frac{2-\ell}{\ell}\Big) \max_S |p|.$$

□

The proof for $n \geq 2$ is reduced to the case $n = 1$ by means of the following geometric result.

Let x_0 be an interior point of the body V and $0 < \ell \leq 1$. Let R stand for a ray emanating from x_0. We set

$$\gamma(\ell) := \sup_S \left(\inf_R \frac{|V \cap R|_1}{|S \cap R|_1} \right),$$

where S runs over all subsets of V of relative measure $\frac{|S|_n}{|V|_n} \geq \ell$. (Here $|\cdot|_1$ is the linear measure on R.)

LEMMA 3.2. *The following is true:*

$$\gamma(\ell) = \frac{1}{1 - \sqrt[n]{1-\ell}}.$$

(For the proof see [11].)

In applications, the following inequality is of common use.

$$T_k\left(\frac{1 + \sqrt[n]{1-\ell}}{1 - \sqrt[n]{1-\ell}}\right) \leq \left(\frac{4n}{\ell}\right)^k. \tag{3.3}$$

Y. Yomdin [28] observed that an inequality similar to (3.1) is valid also for restrictions of polynomials to some discrete subsets of the unit cube in $\mathbb{R}^n$; here one replaces $|S|_n$ by the *metric* (k,n)*-span* of S determined in terms of Vitushkin's bounds for covering numbers of sublevel sets of polynomials of degree k.

3.2. Inequality for plurisubharmonic functions A real-valued function f defined on a domain $\Omega \subset \mathbb{C}^n$ is called *plurisubharmonic* in Ω if f is upper semicontinuous and its restriction to connected components of a complex line intersecting with Ω is subharmonic.

A plurisubharmonic function $f : B_c(0,r) \to \mathbb{R} \cup \{-\infty\}$ belongs to the class $\mathcal{F}_r$ $(r > 1)$ if it satisfies

(i)
$$\sup_{B_c(0,r)} f = 0;$$

(ii)
$$\sup_{B_c(0,1)} f \geq -1.$$

Hereafter $B(x,\rho)$ and $B_c(x,\rho)$ denote the Euclidean ball with center x and radius ρ in $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively.

Let the ball $B(x,t)$ satisfy

$$B(x,t) \subset B_c(x,at) \subset B_c(0,1), \tag{3.4}$$

where $a > 1$ is a fixed constant.

We set

$$E(s) := s + \sqrt{s^2 - 1}, \qquad s \geq 1.$$

Theorem 3.3 (see [7], Th. 1.2). *There is a constant $c = c(a, r) > 0$ such that the inequality*

$$\sup_{B(x,t)} f \leq c \cdot \ln E\left(\frac{1 + \sqrt[n]{1-\ell}}{1 - \sqrt[n]{1-\ell}}\right) + \sup_S f, \qquad \ell := \frac{|S|_n}{|B(x,t)|_n},$$

holds for every $f \in \mathcal{F}_r$ and every measurable subset $S \subset B(x,t)$.

In applications, one uses the inequality

$$E\left(\frac{1 + \sqrt[n]{1-\ell}}{1 - \sqrt[n]{1-\ell}}\right) < \frac{4n}{\ell}. \tag{3.5}$$

The proof of Theorem 3.3 is based on Lemma 3.2, Theorem 2.3 and Theorem 3.1 with $n = 1$.

4. Applications In this section we present several applications of Cartan-Remez type inequalities.

4.1. BMO-property of analytic functions A closed set $S \subset \mathbb{R}^n$ is called (Ahlfors) *k-regular* ($0 \leq k \leq n$) if for every ball $B(x,r)$ with $x \in S$ and $0 < r < \operatorname{diam} S$

$$c_0 r^k \leq \mathcal{H}_k(B_S(x,r)) \leq c_1 r^k,$$

where $B_S(x,r) := S \cap B(x,r)$, $c_0, c_1 > 0$ are constants independent of x and r and $\mathcal{H}_k$ stands for the Hausdorff k-measure on $\mathbb{R}^n$.

The class of k-regular sets, in particular, contains compact Lipschitz k-manifolds (with integer k), Cantor type sets and self-similar sets (with arbitrary k).

Suppose that $S \subset \mathbb{R}^n$ is compact k-regular with $n-1 < k \leq n$. Let $f = \{f_v : v \in V\}$ be a family of real analytic functions defined in a neighbourhood of S depending analytically on a parameter v varying in an open set $V \subset \mathbb{R}^m$.

Theorem 4.1 (see [9], Cor. 2.3). *For each $K \Subset V$ there exists a positive constant $C = C(K, S, f)$ such that for all $f_v \not\equiv 0$, $v \in K$, and $x \in S$, $0 < r < \operatorname{diam} S$,*

$$\frac{1}{\mathcal{H}_k(B_S(x,r))} \int_{B_S(x,r)} \left| \ln \frac{|f_v(y)|}{\sup_{B_S(x,r)} |f_v|} \right| d\mathcal{H}_k(y) \leq C.$$

In particular, $\ln |f_v| \in BMO(S)$ defined by means of $\mathcal{H}_k$ with BMO norm bounded by $2C$.

This result follows from (uniform) Remez type inequalities for the functions of family f, see [9, Th. 2.1], proved with the help of a Cartan-type lemma (cf. subsections 2.2, 2.3 above). Similarly, using Theorem 3.3, one shows that for each not identically $-\infty$ plurisubharmonic function on $B_c(0,r) \Subset \mathbb{C}^n$ its restriction to $B(0,r') \Subset \mathbb{R}^n$, $0<r'<r$, belongs to $BMO(B(0,r'))$ (see [7]).

4.2. L^q norm inequalities for analytic functions A function $\psi : \mathbb{R}^n \to \mathbb{R}_+$ is called *log-concave* if its support $\operatorname{supp}\psi := \{x \in \mathbb{R}^n \,:\, \psi(x) > 0\}$ is convex and $\ln\psi$ is a concave function on $\operatorname{supp}\psi$. Every nonnegative function that is concave over a convex domain is log-concave (its value is assumed to be zero outside the original domain). In particular, the indicator function of a convex body is log-concave.

Let $S \subset \mathbb{R}^n$ be a convex body and ψ be log-concave with support in S. For $1 \le p \le \infty$ we introduce L^p spaces of integrable functions $f : S \to \mathbb{R}$ with norm

$$\|f\|_{p,\psi} := \left(\frac{1}{\int_S \psi(x)dx} \cdot \int_S |f(x)|^p \psi(x)dx\right)^{\frac{1}{p}};$$

here dx is the Euclidean volume form on $\mathbb{R}^n$.

Clearly, if $p<q$, then $\|f\|_{p,\psi} \le \|f\|_{q,\psi}$. It is known for awhile that if f is a polynomial of degree d, then the converse inequality

$$\|f\|_{q,\psi} \le c_{p,q,d}\|f\|_{p,\psi} \tag{4.1}$$

holds with a constant $c_{p,q,d}$ depending on p,q and d only, see, e.g., [12], [15], [20], [23], [25] and references therein. According to [15, p. 234], if one seeks an inequality valid for arbitrary convex bodies K and log-concave functions ψ, then the optimal constant $c_{p,q,d} = \left(\frac{cq}{p}\right)^d$ for an absolute (unspecified) constant $c \ge 1$. An extension of this result to the case of analytic functions is given in [10].

Let $f = \{f_v \,:\, v \in V\}$ be a family of real analytic functions defined on an open set $U \subset \mathbb{R}^n$ depending analytically on a parameter v varying in an open set $V \subset \mathbb{R}^m$. Fix open sets $K_1 \Subset U$ and $K_2 \Subset V$.

Theorem 4.2 (cf. [10], Th. 2.1). *There exists a constant $C = C(K_1,K_2,f) \ge 1$ such that for all convex bodies $S \subset K_1$, log-concave functions ψ with support in S, functions f_v, $v \in K$, and all $1 \le p \le q < \infty$,*

$$\|f_v\|_{q,\psi} \le c_{p,q} \cdot C^{2\left(\frac{1}{p}-\frac{1}{q}\right)} \cdot \left(\frac{4q}{p}\right)^C \|f_v\|_{p,\psi}, \tag{4.2}$$

where

$$c_{p,q} = e^{\frac{2}{p}} \cdot (2(p+1))^{\frac{1}{p}-\frac{1}{q}} \cdot \left(\frac{4q}{p}\right)^{\frac{1}{q}}.$$

If f is the space of real polynomials of degree d on $\mathbb{R}^n$, then $C=d$.

REMARK 4.3. The constant on the right-hand side of (4.2) is bounded from above by $\left(\frac{C_p q}{p}\right)^C$ with $C_p = 4\cdot(8\cdot e^{2+\frac{2}{e}}\cdot(p+1))^{\frac{1}{p}}$ (e.g., $C_p < 4.4$ for $p \geq 100$). In particular, for the optimal constant $c_{p,q,d}$ in (4.1) Theorem 4.2 and an example in [15, p. 234] imply

$$1 \leq \limsup_{p\to\infty} \frac{p\cdot(c_{p,q,d})^{\frac{1}{d}}}{q} \leq 4.$$

It might be of interest to find the precise value of the limit.

The proof of Theorem 4.2 is based on (uniform) Remez type inequalities for functions of family f and on the result of Kannan, Lovász and Simonovits [23, Th. 2.7].

4.3. Jensen-type inequality for families of holomorphic functions Let $f := \{f_v : v \in B_c(x,R)\}$, $R > 1$, be a family of holomorphic in the open unit disk $\mathbb{D}$ functions depending holomorphically on a parameter v varying in the open Euclidean ball $B_c(x,R) \subset \mathbb{C}^s$. By $\mathcal{N}_{f,\rho}(v)$ we denote the number of zeros (counting with multiplicities) of f_v in the closed disk $\bar{\mathbb{D}}_\rho$, $\rho < 1$. (We set $\mathcal{N}_{f,\rho}(v) = -\infty$ if $f_v \equiv 0$). It is known that the function $\mathcal{N}_{f,\rho}$ is Borel measurable on $B_c(x,R)$.

Let v_0 be the *valency* of $f_0(v) := f_v(0)$, $v \in \bar{B}_c\big(x, \frac{R+1}{2}\big)$, that is, the supremum of valencies of restrictions of f_0 to complex straight lines passing through points of $\bar{B}_c\big(x, \frac{R+1}{2}\big)$.[1]

THEOREM 4.4 (see [8], Th.1.5, Cor.1.6). *Let $V \subset B(x,1)$ be a convex set of real dimension k. Suppose that*

$$\sup_{v\in V}\sup_{z\in\mathbb{D}} |f_v(z)| \leq M_1 < \infty \qquad \text{and} \qquad \sup_{v\in V} |f_0(v)| \geq M_2 > 0. \tag{4.3}$$

There exists a constant $c = c(R) > 0$ such that for $\rho < 1$ and every $T \geq 0$,

$$|\{v \in V : \mathcal{N}_{f,\rho} \geq T\}|_k \leq 4k\left(\frac{M_1}{M_2}\right)^{\frac{1}{cv_0}} \cdot e^{-\left(\frac{\ln(1/\rho)}{cv_0}\right)T} \cdot |V|_k. \tag{4.4}$$

In particular,

$$\frac{1}{|V|_k}\int_V \mathcal{N}_{f,\rho}(v)dv \leq \frac{cv_0\ln(4ek) + \ln(M_1/M_2)}{\ln(1/\rho)}. \tag{4.5}$$

(Here $|\cdot|_k$ stands for Lebesgue k-measure on the k-dimensional affine subspace in $\mathbb{R}^s$ containing V.)

The following example illustrates one of the possible applications of Theorem 4.4.

[1] Recall that the *valency* of a holomorphic function $h : \mathbb{D} \to \mathbb{C}$ in $\bar{\mathbb{D}}_\rho$, $\rho < 1$, is the supremum of numbers of zeros (counting with multiplicities) in $\bar{\mathbb{D}}_\rho$ of functions $h - c$ for c running through $\mathbb{C}$.

EXAMPLE 4.5. The second part of Hilbert's sixteenth problem asks whether the number of isolated closed trajectories (*limit cycles*) of a planar polynomial vector field is always bounded in terms of its degree. This is the most prominent finiteness problem, related to a fairly general class of algebraic differential equations and is one of the few Hilbert's problems which remain unsolved. Recent results of Y. Il'yashenko [22] and of Écalle, Martinet, Moussu and Ramis [17] give global finiteness of the number of limit cycles for each individual vector field (but leave open the question of the existence of a bound depending on the degree only). Below we consider the local version of the problem in which one asks for explicit bounds (in terms of degree only) on the number of limit cycles situated in a small neighbourhood of a singular point of the vector field. Except for the result of Bautin [2], the answer to this problem is not known. According to Smale (see [27]) the global estimate should be polynomial in the degree d of the components of the vector field. Our estimate below (local case) is "in the mean" but gives a substantially better bound of the order $\ln d$. Moreover, it is known that locally the number of limit cycles can be about $\frac{d^2}{2}$ which is essentially larger than $\ln d$. We now formulate our result precisely.

Consider the planar polynomial vector field of degree d

$$\dot{x} = -y + \sum_{1\le i+k\le d} a_{ki}x^k y^i, \qquad \dot{y} = x + \sum_{1\le i+k\le d} b_{ki}x^k y^i; \tag{4.6}$$
$$\sum_{i,k} |a_{ki}|^2 + \sum_{i,k} |b_{ki}|^2 \le N^2.$$

Let $s := d(d+3)$, $d \ge 2$, be dimension of the space E_d of real coefficients of these fields. We naturally identify E_d with $\mathbb{R}^s$. For $v \in B(0,N) \subset E_d$ by $C(v)$ we denote the number of limit cycles of the corresponding field $\mathcal{F}(v)$ in the open disk $D_{\frac{1}{2}} \subset \mathbb{R}^2$ of radius $\frac{1}{2}$ with center at 0. By $v_l = (a_{10}, a_{01}, b_{10}, b_{01}) \in \mathbb{R}^4$ we denote the tuple of coefficients of the linear part of $\mathcal{F}(v)$.

Consider the projection $\pi : \mathbb{R}^s \to \mathbb{R}^4$, $v \mapsto v_l$.

THEOREM 4.6 (see [8], Th.1.3). *Let $N \le \frac{1}{40\pi\sqrt{d}}$ and $V \subset B(0,N) \subset E_d$ be a convex set of dimension k such that*

$$\frac{|\pi(B(0,N))|_4}{|\pi(V)|_4} \le \delta < \infty. \tag{4.7}$$

There exist absolute constants $c_1, c_2, c_3 > 0$ such that for any $T \ge 0$

$$|\{v \in V \,:\, C(v) \ge T\}|_k \le c_1 \delta d^{c_2} e^{-c_3 T} |V|_k. \tag{4.8}$$

In particular, there is an absolute constant $c > 0$ such that

$$\frac{1}{|V|_k} \int_V C(v)\, dv \le c(\ln\delta + \ln d). \tag{4.9}$$

In view of Smale's conjecture, the above inequalities are especially interesting for δ depending polynomially on d.

4.4. Counting zeros of holomorphic polynomials along complex curves in $\mathbb{C}^2$
Let $f : \mathbb{C} \to \mathbb{C}$ be a nonpolynomial entire function. Let $\mathcal{P}_d$ be the space of holomorphic polynomials of degree d on $\mathbb{C}^2$. For $p \in \mathcal{P}_d \setminus \{0\}$ by $n_{p_f}(r)$, $r > 0$, we denote the number of zeros (counting with multiplicities) of the function $p_f(z) := p(z, f(z))$, $z \in \mathbb{D}_r$. We set

$$\mathcal{N}_{f;d}(r) := \sup_{p \in \mathcal{P}_d \setminus \{0\}} n_{p_f}(r).$$

THEOREM 4.7 (see [6], Th. 2.5). *There exists a (depending on f) sequence of natural numbers $\{d_j\}$ convergent to ∞ such that for each $r > 0$*

$$\lim_{j \to \infty} \frac{\ln \mathcal{N}_{f;d_j}(r)}{\ln d_j} = 2. \tag{4.10}$$

REMARK 4.8. In many cases (e.g., for f being a "generalized" exponential polynomial), the above sequence $\{d_j\}$ consists of all natural numbers, see [6] for details.

One also proves Bernstein, Markov and Remez type inequalities for families of functions $\{p_f : p \in \mathcal{P}_d, d \in \mathbb{Z}_+\}$, see [6]. Some particular cases of these results were earlier obtained in [14].

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Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, T2N 1N4
e-mail: albru@math.ucalgary.ca