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Divergent Series: Past, Present, Future ...

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Abstract. The paper presents some reflections of the author on divergent series and their role and place in mathematics over the centuries. The point of view presented here is limited to differential equations and dynamical systems.

Résumé. L'article présente quelques réflexions de l'auteure sur les séries divergentes, et leur rôle et place en mathématiques au courant des siècles. Le point de vue présenté ici est limité aux équations différentielles et aux systèmes dynamiques.

1. Introduction In the past, divergent series have played a central role in mathematics. Mathematicians like, for instance, Lacroix, Fourier, Euler, Laplace, etc. have used them extensively. Nowadays, they play a marginal role in mathematics and are often not mentioned in the standard curriculum. A significant number of mathematicians and most students are not aware that they can be of any use.

Where was the turn?

According to [2], the turn occurred at the time of Cauchy and Abel, when the need was felt for constructing analysis on absolute rigor.

So let us go back and see what Cauchy and Abel have said of divergent series.

Cauchy (Preface of "Analyse mathématique", 1821): *"J'ai été forcé d'admettre diverses propositions qui paraîtront peut-être un peu dures. Par exemple qu'une série divergente n'a pas de somme..."* ("I have been forced to admit some propositions which will seem, perhaps, hard to accept. For instance that a divergent series has no sum ...")

Cauchy made one exception: he justified rigorously the use of the divergent Stirling series to calculate the Γ-function. We will explain below the kind of argument he made when looking at the example of the Euler differential equation.

Abel (Letter to Holmboe, January 16, 1826): *"Les séries divergentes sont en général quelque chose de bien fatal et c'est une honte qu'on ose y fonder*

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aucune démonstration. On peut démontrer tout ce qu'on veut en les employant, et ce sont elles qui ont fait tant de malheurs et qui ont enfanté tant de paradoxes. ... Enfin mes yeux se sont dessillés d'une manière frappante, car à l'exception des cas les plus simples, par exemple les séries géométriques, il ne se trouve dans les mathématiques presque aucune série infinie dont la somme soit déterminée d'une manière rigoureuse, c'est-à-dire que la partie la plus essentielle des mathématiques est sans fondement. Pour la plus grande partie les résultats sont justes il est vrai, mais c'est là une chose bien étrange. Je m'occupe à en chercher la raison, problème très intéressant." ("Divergent series are, in general, something terrible and it is a shame to base any proof on them. We can prove anything by using them and they have caused so much misery and created so many paradoxes. .... Finally my eyes were suddenly opened since, with the exception of the simplest cases, for instance the geometric series, we hardly find, in mathematics, any infinite series whose sum may be determined in a rigorous fashion, which means the most essential part of mathematics has no foundation. For the most part, it is true that the results are correct, which is very strange. I am working to find out why, a very interesting problem.")

The author was struck by this citation when she first read it 26 years ago. At that time she knew nothing about divergent series. But this citation has followed her for her whole career. In her point of view, this citation contains the past, present and future of divergent series in mathematics.

The past: As remarked by Abel, divergent series occur very often in many natural problems of mathematics and physics. Their use has permitted to do successfully a lot of numerical calculations. One example of this is the computation by Laplace of the secular perturbation of the orbit of the Earth around the Sun due to the attraction of Jupiter. The calculations of Laplace are verified experimentally, although the series he used were divergent.

The present: In the 20-th century divergent series have occupied a marginal place. However it is during the same period that we have learnt to justify rigorously their use, answering at least partially Abel's question. In the context of differential equations, the Borel summation, generalized by Écalle and others to multi-summability, gives good results (see for instance [1], [2], [3]).

The future: Why do so many important problems of mathematics lead to divergent series (see for instance [4])? What is the meaning of a series being divergent? Finding an answer to this question is a fascinating research field.

We will illustrate all this on the example of the Euler differential equation:

$$x^2y' + y = x. \tag{1}$$

Since this is a short paper, the list of references is by no means exhaustive.

2. The Past The Euler differential equation (1) has the formal solution

$$\hat{f}(x) = \sum_{n\geq 0}(-1)^n n! x^{n+1}, \tag{2}$$

which is divergent for all nonzero values of x.

On the other hand, it is a linear differential equation whose solution can be found by quadrature:

$$f(x) = e^{\frac{1}{x}} \int_0^x \frac{e^{-\frac{1}{z}}}{z} dz. \tag{3}$$

The integral is convergent for $x > 0$ and hence yields a solution of (1). We can rewrite this solution as

$$f(x) = \int_0^x \frac{e^{\frac{1}{x}-\frac{1}{z}}}{z} dz, \tag{4}$$

in which we make the change of coordinate $\frac{\zeta}{x} = \frac{1}{z} - \frac{1}{x}$. This yields

$$f(x) = \int_0^{+\infty} \frac{e^{-\frac{\zeta}{x}}}{1+\zeta} d\zeta. \tag{5}$$

The integral is convergent for $x \geq 0$. Hence, the solution $f(x)$ is well defined for $x > 0$ and, moreover, $\lim_{x\to 0^+} f(x) = 0$.

What is now the link between the divergent power series $\hat{f}(x)$ and the function $f(x)$? We will show that the difference between $f(x)$ and a partial sum

$$f_k(x) = \sum_{n=0}^{k-1}(-1)^n n! x^{n+1} \tag{6}$$

of the series is smaller than the first neglected term (this part has been inspired by [6]). This is exactly as in the Leibniz criterion for alternating series.

PROPOSITION 2.1. *For any $x \geq 0$*

$$|f(x) - f_k(x)| \leq k! x^{k+1}. \tag{7}$$

PROOF. We use the following formula which is easily checked

$$\frac{1}{1+\zeta} = \sum_{n=0}^{k-1}(-1)^n \zeta^n + (-1)^k \frac{\zeta^k}{1+\zeta}. \tag{8}$$

Then

$$
\begin{aligned}
f(x) &= \int_0^{+\infty} e^{-\frac{\zeta}{x}} \left(\sum_{n=0}^{k-1} (-1)^n \zeta^n + (-1)^k \frac{\zeta^k}{1+\zeta} \right) d\zeta \\
&= \sum_{n=0}^{k-1} \int_0^{+\infty} (-1)^n \zeta^n e^{-\frac{\zeta}{x}} d\zeta + \int_0^{+\infty} (-1)^k \frac{\zeta^k e^{-\frac{\zeta}{x}}}{1+\zeta} d\zeta.
\end{aligned} \tag{9}
$$

Using that

$$
\Gamma(n+1) = n! = \int_0^{\infty} z^n e^{-z} dz, \tag{10}
$$

which implies

$$
\int_0^{\infty} \zeta^n e^{-\frac{\zeta}{x}} d\zeta = n! x^{n+1}, \tag{11}
$$

this yields

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{k-1} (-1)^n n! x^{n+1} + \int_0^{+\infty} (-1)^k \frac{\zeta^k e^{-\frac{\zeta}{x}}}{1+\zeta} d\zeta \\
&= f_k(x) + R_k(x),
\end{aligned} \tag{12}
$$

where $R_k(x)$, the remainder, is the difference between $f(x)$ and the partial sum $f_k(x)$ of the power series. The result follows since

$$
|R_k(x)| = \int_0^{\infty} \frac{\zeta^k e^{-\frac{\zeta}{x}}}{1+\zeta} d\zeta \leq \int_0^{\infty} \zeta^k e^{-\frac{\zeta}{x}} d\zeta = k! x^{k+1}. \tag{13}
$$

□

The argument given here, which justifies rigorously the use of the divergent series in numerical calculations, is very similar to the argument made by Cauchy for the use of the Stirling series. In particular, Cauchy used the formula (8).

If we use the power series to approximate the function $f(x)$, how good is the approximation?

We encounter here the main difference between convergent and divergent series. With convergent series, the more terms we take in the partial sum, the better the approximation. This is no more the case with divergent series. Indeed, if we take x fixed in (2) the general term tends to ∞. Hence, we are better to take the partial sum for which the first neglected term is minimum. In our example, $k!x^{k+1}$ is minimum when $x \sim \frac{1}{k}$, in which case $\frac{1}{x} \sim k$. We use Stirling formula to approximate $k!$ for k large:

$$k! = k(k-1)! = k\Gamma(k) \sim \sqrt{2\pi}k^{k+\frac{1}{2}}e^{-k}. \tag{14}$$

This gives us

$$|R_k(x)| \leq \sqrt{2\pi}\frac{e^{-k}}{\sqrt{k}} \sim \sqrt{2\pi x}e^{-\frac{1}{x}}, \tag{15}$$

which is exponentially small for x small. Not only have we bounded the error made when approximating the function by the partial sum of the power series, but this error is exponentially small for small x, i.e. very satisfactory from the numerical point of view.

The phenomenon we have described here is not isolated and explains the successes encountered when using divergent series in numerical approximations.

3. The Present Looking at what we have done with the Euler equation, someone can have the impression that we have cheated. Indeed, we have chosen a linear differential equation, thus allowing us to construct by quadrature a function which is a solution of the differential equation. But what about the general case?

In general, once we have a formal solution by means of a power series, we use a *theory of resummation* for finding a function which is a solution. In this paper, we will focus on the *Borel method of resummation*, also called 1-summability. We start with the properties that an adequate theory of summability must satisfy:

Properties of a good method of resummation (see for instance [2]): We consider a series $\sum_{n=0}^{\infty} a_n$, to which we want to associate a number S, called its sum:

(1) If $\sum_{n=0}^{\infty} a_n$ is convergent, then S should be the usual sum of the series.
(2) If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are summable with respective sums S and S' then $\sum_{n=0}^{\infty}(a_n + Cb_n)$ is summable with sum $S + CS'$.
(3) A series $\sum_{n=0}^{\infty} a_n$ is *absolutely summable* if the series $\sum_{n=0}^{\infty} |a_n|$ is summable.
(4) If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely summable with respective sums S and S', then the product of the series, $\sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$, is absolutely summable with sum SS'.
(5) If $\sum_{n=0}^{\infty} a_n$ is summable with sum S, then $\sum_{n=1}^{\infty} a_n$ is summable with sum $S - a_0$.
(6) (In the context of differential equations) If $\sum_{n=0}^{\infty} a_n x^n$ is summable with sum $f(x)$, then $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is summable with sum $f'(x)$.

The Borel method of resummation for a divergent power series: We present it in a way which proves at the same time the property (1): the idea is to take a convergent power series $\sum_{n=0}^{\infty} a_n x^{n+1}$ and to write its sum $S(x) =$

$\lim_{k\to\infty}\sum_{n=0}^{k} a_n x^{n+1}$ in a different way. For this purpose, we use (10) and we write

$$\begin{aligned} S(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n!}\, n! \\ &= \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n!} \int_0^{\infty} z^n e^{-z} dz \\ &= \int_0^{\infty} \left(\sum_{n=0}^{\infty} \frac{a_n (xz)^n}{n!}\right) e^{-z} x dz \\ &= \int_0^{\infty} \left(\sum_{n=0}^{\infty} \frac{a_n \zeta^n}{n!}\right) e^{-\frac{\zeta}{x}} d\zeta. \end{aligned} \tag{16}$$

Changing the order of the summation and the integral in the last line requires a proof, which we leave as an exercise.

This suggests the following definition, which is well adapted to further analytic extension.

DEFINITION 3.1. 1. A power series $\sum_{n=0}^{\infty} a_n x^{n+1}$ is 1-summable (Borel-summable) in the direction d, where d is a half-line from the origin in the complex plane if

- the series $\sum_{n=0}^{\infty} a_n \frac{\zeta^n}{n!}$ has a nonzero radius of convergence, and the sum of the series is an analytic function $g(\zeta)$ on the disk of convergence,
- the function $g(\zeta)$ can be extended along the half line d,
- and the integral

$$\int_d g(\zeta) e^{-\frac{\zeta}{x}} d\zeta \tag{17}$$

is convergent in some domain, with value $S(x)$.

We call $S(x)$ the sum of the power series.

2. A power series $\sum_{n=0}^{\infty} a_n x^{n+1}$ is 1-summable if it is 1-summable in all directions d, except a finite number of exceptional directions.

Example: In the case of the solution to Euler's differential equation we have $a_n = (-1)^n n!$. Then

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} \zeta^n = \frac{1}{1+\zeta} = g(\zeta).$$

Hence,

$$S(x) = \int_0^{\infty} g(\zeta) e^{-\frac{\zeta}{z}} dz = \int_0^{\infty} \frac{e^{-\frac{\zeta}{x}}}{1+\zeta} d\zeta, \tag{18}$$

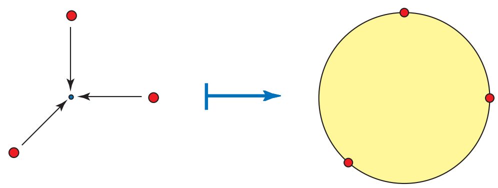


Figure 1: The Borel transforms sends the hidden singularities to a finite distance.

which is precisely (5). The solution of Euler's differential equation is 1-summable in all directions except the direction $\mathbb{R}^-$. The problem with the direction $\mathbb{R}^-$ comes from the singularity at $\zeta = -1$ in (5) or (18).

A very powerful insight coming from Borel. It is well known that if a power series $\sum_{n=0}^{\infty} a_n x^{n+1}$ has a radius of convergence equal to r and its sum is a function $f(x)$ for $|x| < r$, then the function $f(x)$ has a least one singularity on the circle $|x| = r$. The idea of Borel is that a divergent series is a series with radius of convergence $r = 0$. Hence, we have at least one singularity hidden in some direction: for Euler's differential equation, this is the direction $\mathbb{R}^-$. The operation

$$\hat{f}(x) = \sum_{n=0}^{\infty} a_n x^{n+1} \qquad \mapsto \qquad \mathcal{B}(\hat{f}) = \sum_{n=0}^{\infty} \frac{a_n \zeta^n}{n!}, \tag{19}$$

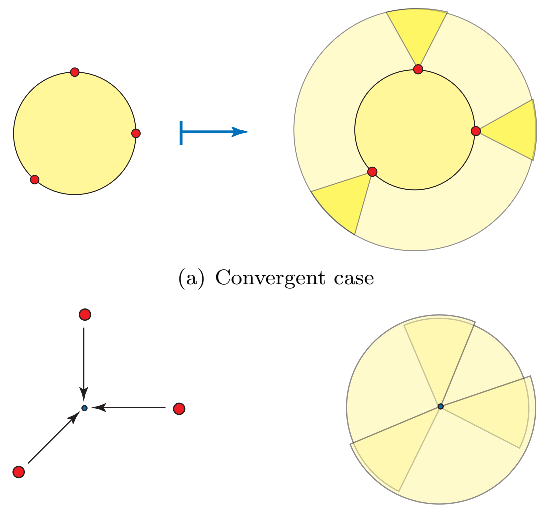
sends the singularities at a finite distance, as if we had blown the disk of convergence to bring it from radius 0 to a finite radius r (see Figure 1). $\mathcal{B}(\hat{f})$ is called the *Borel transform* of $\hat{f}$. For a convergent power series on a disk, we can extend the function outside the disk of convergence, as long as we avoid the singularities. In particular, we can turn around the singularities (see Figure 2(a)). Generically, the extension is ramified at the singularities. If we have only a finite number of singularities and we now shrink the radius of the disk to 0, then we get a function defined on a finite number of sectors covering a pointed neighborhood of the origin (see Figure 2(b)), which is what is provided by the Borel summability.

A theory of resummation is useful if there are theorems associated to it. For instance, for the Borel summation, let us cite this theorem of Borel, which has been followed by many other theorems of the same type.

Theorem 3.2. *[2] We consider an algebraic differential equation*

$$F(x, y, y', \dots y^{(m)}) = 0, \tag{20}$$

where F is a multivariate polynomial. If $\hat{f}(x) = \sum_{n=0}^{\infty} a_n x^n$ is a formal solution of (20) *and $\hat{f}$ is absolutely Borel-summable with Borel sum $f(x)$, then $f(x)$ is solution of the differential equation* (20) *and has the asymptotic expansion $\hat{f}(x) = \sum_{n=0}^{\infty} a_n x^n$.*



(a) Convergent case

(b) Divergent case

Figure 2: We can extend the function around the singularities.

REMARK 3.3. (1) The sums of a 1-summable series in the different directions d give functions which are analytic extensions one of the other as long as we move the line d continuously through directions in which the series is 1-summable. This yields a function defined on a sector with vertex at the origin. More details can be found for instance in [1], [5] and [6].
(2) The Borel sum of a divergent power series can never be uniform in a punctured neighborhood of the origin. It is necessarily ramified. This is what is known in the literature as the *Stokes phenomenon*.

The 1-summability and its extensions have been extensively studied during the 20-th century. Extensions of the notion of 1-summability have been obtained by allowing ramifications $x = \xi^m$. Then 1-summability in x corresponds to m-summability in ξ. The notion of multi-summability has also been introduced: a series $\hat{f}$ is multi-summable if it is a finite sum $\hat{f} = \hat{f}_1 + \cdots + \hat{f}_n$, each $\hat{f}_i$ being m_i-summable. Explicit criteria allow deciding a priori that some systems of differential equations have multi-summable solutions in the neighborhood of certain singular points (see for instance [1]).

4. **The Future** Let us now look at a generalized Euler differential equation

$$x^2 y' + y = g(x). \tag{21}$$

For almost all functions $g(x)$ analytic in the neighborhood of the origin and such that $g(0) = 0$ the formal solution of (21) vanishing at the origin is given by a divergent series. Only in very special cases the equation (21) has an analytic solution at the origin. For instance, $f(x) = x$ is the analytic solution of

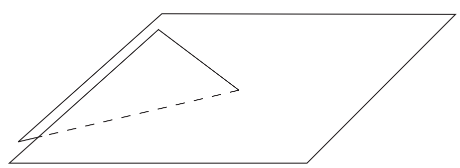
$$x^2y' + y = x + x^2. \tag{22}$$

What is the difference between the equations (1) and (22)? To understand we apply successively the two steps:

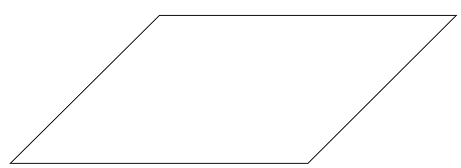
- Complexification: we allow $x \in \mathbb{C}$;
- Unfolding: $x = 0$ is a double singular point of each equation. Hence, we introduce a parameter so as to split the double singular point into two simple singular points.

Complexification: let us consider a function $f(x)$ which is the Borel sum of a solution of (21), and its analytic extension when we turn around the origin. The function $f(x)$ is (Figure 3):

- uniform for (22);
- ramified for (1), and generically ramified for a solution of (21). The two branches differ by a multiple of $e^{\frac{1}{x}}$ (which is a solution of the homogeneous equation).



(a) Equation (1)



(b) Equation (22)

Figure 3: The domain of $f(x)$

How to understand that generically we should have ramification? To answer, we unfold and embed (21) into

$$(x^2 - \epsilon)y' + y = g_\epsilon(x). \tag{23}$$

We limit here our discussion to $\epsilon > 0$, although all complex values of ϵ are of interest. The solutions of the homogeneous equation associated to (23) are given by $C\left(\frac{x+\sqrt{\epsilon}}{x-\sqrt{\epsilon}}\right)^{\frac{1}{2\sqrt{\epsilon}}}$. We localize at the singular points using the changes of coordinate $X_\pm = x \mp \sqrt{\epsilon}$, thus transforming the equation to the form

$$X_\pm y' + \frac{y}{X_\pm \pm 2\sqrt{\epsilon}} = k_\epsilon^\pm(X_\pm).$$

A particular solution $y = h_1(x)$ near $x_1 = \sqrt{\epsilon}$ is analytic. Indeed, taking $X_+ = x - \sqrt{\epsilon}$, it is given by

$$h_1(x) = \left(\frac{X_+ + 2\sqrt{\epsilon}}{X_+}\right)^{\frac{1}{2\sqrt{\epsilon}}} \int_0^{X_+} k_\epsilon^+(z) z^{\frac{1}{2\sqrt{\epsilon}}-1}(z+2\sqrt{\epsilon})^{-\frac{1}{2\sqrt{\epsilon}}}\,dz.$$

The general solution is given by

$$y(x) = h_1(x) + C_1\left(\frac{x+\sqrt{\epsilon}}{x-\sqrt{\epsilon}}\right)^{\frac{1}{2\sqrt{\epsilon}}}.$$

Hence, the solution $h_1(x)$ (corresponding to $C_1 = 0$) is the unique solution which is analytic and bounded at $x = \sqrt{\epsilon}$.

Similarly, if we let $X_- = x + \sqrt{\epsilon}$, a particular solution $y = h_2(x)$ near $x_2 = -\sqrt{\epsilon}$ is given by

$$h_2(x) = \left(\frac{X_-}{X_- - 2\sqrt{\epsilon}}\right)^{\frac{1}{2\sqrt{\epsilon}}} \int_0^{X_-} k_\epsilon^-(z) z^{-\frac{1}{2\sqrt{\epsilon}}-1}(z-2\sqrt{\epsilon})^{\frac{1}{2\sqrt{\epsilon}}}\,dz.$$

The general solution is also given by

$$y(x) = h_2(x) + C_2\left(\frac{x+\sqrt{\epsilon}}{x-\sqrt{\epsilon}}\right)^{\frac{1}{2\sqrt{\epsilon}}}.$$

We now have two cases:

(1) If $\frac{1}{2\sqrt{\epsilon}} \notin \mathbb{N}$, then h_2 is analytic near $x = -\sqrt{\epsilon}$. Since $C_2\left(\frac{x+\sqrt{\epsilon}}{x-\sqrt{\epsilon}}\right)^{\frac{1}{2\sqrt{\epsilon}}}$ is ramified at $x = -\sqrt{\epsilon}$ for $C_2 \neq 0$, all solutions but h_2 are ramified. It is of course exceptional that the extension of h_1 at $-\sqrt{\epsilon}$ be exactly the solution h_2 and, generically, we should expect that the analytic extension of $h_1(x)$ is $h_2(x) + C_2\left(\dfrac{x+\sqrt{\epsilon}}{x-\sqrt{\epsilon}}\right)^{\frac{1}{2\sqrt{\epsilon}}}$ with $C_2 \neq 0$. Hence the extension of h_1 should be ramified at $x = -\sqrt{\epsilon}$. If this ramification holds till the limit $\epsilon = 0$ we get Figure 1(a).

(2) Let us now consider the case $\frac{1}{2\sqrt{\epsilon}} = n \in \mathbb{N}$. We must again divide the discussion in two cases:

(i) In the generic case, h_2 is ramified: it contains one term of the form $(x+\sqrt{\epsilon})^n \ln(x+\sqrt{\epsilon})$.

(ii) In the exceptional case, h_2 is analytic, and so are all solutions through $x=-\sqrt{\epsilon}$. Hence, it is impossible for h_1 to be ramified at $x=-\sqrt{\epsilon}$. This case is excluded in the unfolding of an equation (21) whose solution is ramified.

Let us now summarize the very interesting phenomenon we have discovered: if the formal solution of (21) is divergent, then in the unfolding:

- Necessarily $h_1(x)$ is ramified at $-\sqrt{\epsilon}$: the divergence means a form of *incompatibility* between the special local solutions at the two singular points $\pm\sqrt{\epsilon}$, which remains until the limit at $\epsilon = 0$.
- **Parametric resurgence phenomenon:** for sequences of values of the parameter ϵ converging to $\epsilon = 0$ (here $\frac{1}{2\sqrt{\epsilon}} \in \mathbb{N}$), the pathology of the system is located exactly at one of the singular points. Indeed, here, the only way for the system to have an incompatibility is that one of the singular points be pathologic.

We have understood why divergence is the rule, and convergence the exception.

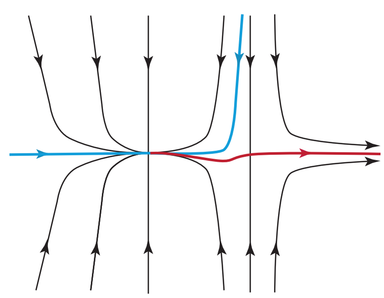


Figure 4: The analytic invariant manifolds of the saddle and the node in (26) do not match.

Application. The generalized Euler differential equation (21) is the equation for the center manifold of the saddle-node in the two-dimensional analytic vector field

(24)
$$\begin{aligned}\dot{x} &= x^2,\\ \dot{y} &= -y + g(x),\end{aligned}$$

with $g(0) = 0$. More generally, if we have a saddle-node in a (real) two-dimensional analytic vector field

$$\begin{aligned} \dot{x} &= x^2, \\ \dot{y} &= -y + O(x^2) + O(y^2). \end{aligned} \tag{25}$$

it is known that, generically, the center-manifold is only $\mathbb{C}^\infty$. Now, we understand why. Indeed, the unfolded system

$$\begin{aligned} \dot{x} &= x^2 - \epsilon, \\ \dot{y} &= -y + O(x^2 - \epsilon) + O(y^2), \end{aligned} \tag{26}$$

has two singular points at $(\pm\sqrt{\epsilon}, 0)$ with ratio of eigenvalues

$$\lambda_\pm(\epsilon) = \pm 2\sqrt{\epsilon}\left(1 + O(\sqrt{\epsilon})\right).$$

When $\epsilon > 0$, the point $(\sqrt{\epsilon}, 0)$ is a saddle, and the point $(-\sqrt{\epsilon}, 0)$ is a node. The saddle has an analytic unstable manifold. For $\epsilon \notin \frac{1}{\mathbb{N}}$, then the node is linearizable. In the linearizing coordinates (X, Y), the trajectories have the form $Y = CX^{\frac{1}{2\sqrt{\epsilon}}}$ and $X = 0$. Hence, the node has exactly two invariant analytic manifolds $X = 0$ and $Y = 0$. There is no reason why $Y = 0$ would match with the unstable manifold of the saddle (see Figures 4 and 5). If it does not match, then, when taking complex coordinates, the unstable manifold of the saddle is ramified at the node. If the mismatch is up to the limit, then we expect the center manifold to be not analytic at the limit, and only 1-summable.

When $\epsilon \in \frac{1}{\mathbb{N}}$, then we have the parametric resurgence phenomenon. If the saddle-node at the limit has no analytic center manifold, then the node is resonant with normal form

$$\begin{aligned} \dot{X} &= -\frac{1}{n}X, \\ \dot{Y} &= -Y + a_n X^n, \end{aligned} \tag{27}$$

with $a_n \neq 0$.

5. **Conclusion** The phenomena described in Section 4 are much more general than the context of (21) discussed here, and are explored by the author and her collaborators in different contexts: parabolic points of 1-dimensional diffeomorphisms, saddle-nodes and resonant saddles of two-dimensional vector fields, Hopf bifurcation in two-dimensional vector fields, non resonant irregular singular points of linear differential systems. A common thread is that 1-summability occurs when two equilibrium points of a dynamical system coallesce in a double point, or more generally two special "invariant objects" (for instance a singular point and a limit cycle) coallesce. There are some rigid dynamics attached to

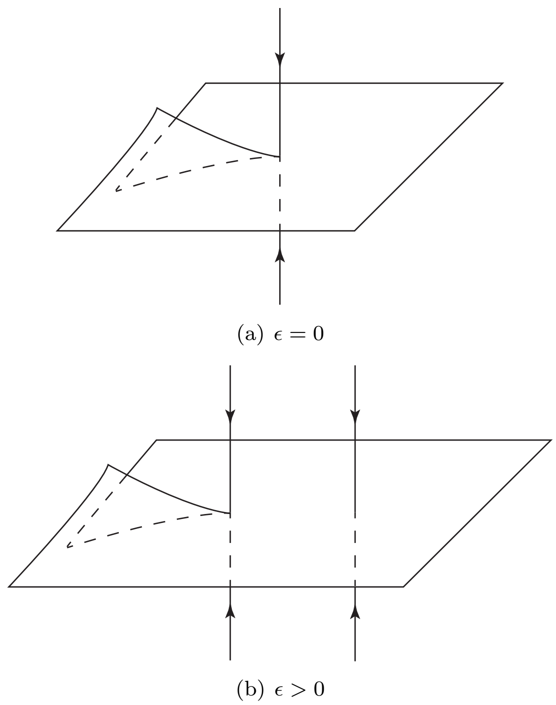


Figure 5: A non analytic center manifold of a saddle-node (26) and its unfolding: the x-coordinate is drawn in $\mathbb{C}$.

each simple equilibrium (object), and these rigid dynamics do not match well until the limit when the two equilibria (or objects) coallesce. More generally, k-summability occurs when $k+1$ equilibrium points (or objects) merge together in a generic way. Multi-summability can occur when the merging is non generic.

Divergent series also occur in the phenomena involving "small denominators". A source of divergence in this case is the accumulation of particular solutions. For instance, in the case of a fixed point of a germ of analytic diffeomorphism $f:(\mathbb{C},0)\to(\mathbb{C},0)$, $f(z)=e^{2\pi i\alpha}z+o(z)$, with $\alpha\in\mathbb{R}\setminus\mathbb{Q}$, the divergence of the linearizing series could come from the accumulation of periodic points around the fixed point at the origin. The study of this phenomenon is part of the work for which Jean-Christophe Yoccoz received the Fields medal in 1994.

Divergent series occur generically in many situations within differential equations and dynamical systems. The divergence of the series carries a lot of geometric information on the solutions. For instance, if the formal power series solution of the Euler equation (1) had been convergent, its sum could not have been ramified.

The Future:

- A large program of research will consist in learning to "read" all the rich behaviour of functions defined by divergent series.
- Shouldn't divergent series occupy a more important place in mathematics?

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