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SHARP MAXIMAL FUNCTION ESTIMATES AND BOUNDEDNESS FOR THE TOEPLITZ TYPE OPERATOR ASSOCIATED TO A MULTIPLIER OPERATOR

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ABSTRACT. In this paper, we establish sharp maximal function estimates for the Toeplitz type operator associated to a certain multiplier operator. As an application, we obtain the boundedness of the operator on Lebesgue, Morrey and Triebel-Lizorkin spaces.

RÉSUMÉ. Dans cet article, on établit des estimations de la fonction maximale optimale pour l'opérateur de type Toeplitz associé à un certain opérateur multiplicateur. Comme application, nous obtenons le caractère borné de l'opérateur sur les espaces de Lebesgue, de Morrey et de Triebel-Lizorkin.

1. Introduction and Preliminaries As part of the development of singular integral operators (see [5][19]), their commutators have been well studied. In [2][18], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [1]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [6][15], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [7][9][11], some Toeplitz type operators associated to singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by BMO and Lipschitz functions are obtained. In this paper, we will study the Toeplitz type operators generated by certain multiplier operators and the Lipschitz and BMO functions.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

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where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)dx$. It is well-known that (see [5][19])

$$M^{\#}(f)(x) \approx \sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q |f(y)-c|dy.$$

We say that f belongs to $BMO(R^n)$ if $M^{\#}(f)$ belongs to $L^\infty(R^n)$ and define $||f||_{BMO} = ||M^{\#}(f)||_{L^\infty}$. It has been known that (see [19])

$$||f - f_{2^kQ}||_{BMO} \le Ck||f||_{BMO}.$$

Let

$$M(f)(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < n$ and $1 \le r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q\ni x}\left(\frac{1}{|Q|^{1-r\eta/n}}\int_Q |f(y)|^r dy\right)^{1/r}.$$

The A_p weight is defined by (see [5])

$$A_p = \left\{ w \in L^1_{loc}(R^n) : \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty \right\},$$

$$1 < p < \infty,$$

and

$$A_1 = \{w \in L^p_{loc}(R^n) : M(w)(x) \le Cw(x), a.e.\}.$$

For $\beta > 0$ and $p > 1$, let $\dot{F}^{\beta,\infty}_p(R^n)$ be the homogeneous Triebel-Lizorkin space(see [15]).

For $\beta > 0$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,y\in R^n\\ x\ne y}}\frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty.$$

Definition 1. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \le D\varphi(t) \quad \text{for} \quad t \ge 0.$$

Let f be a locally integrable function on R^n. Set, for $0 \le \eta < n$ and $1 \le p < n/\eta$,

$$||f||_{L^{p,\eta,\varphi}} = \sup_{x\in R^n,\ d>0}\left(\frac{1}{\varphi(d)^{1-p\eta/n}}\int_{Q(x,d)} |f(y)|^p dy\right)^{1/p},$$

where $Q(x,d)=\{y\in R^n : |x-y|<d\}$. The generalized fractional Morrey space is defined by

$$L^{p,\eta,\varphi}(R^n)=\{f\in L^1_{loc}(R^n) : ||f||_{L^{p,\varphi}}<\infty\}.$$

We write $L^{p,\eta,\varphi}(R^n)=L^{p,\varphi}(R^n)$ if $\eta=0$, which is the generalized Morrey space. If $\varphi(d)=d^\delta$, $\delta>0$, then $L^{p,\varphi}(R^n)=L^{p,\delta}(R^n)$, which is the classical Morrey spaces (see [16][17]). If $\varphi(d)=1$, then $L^{p,\varphi}(R^n)=L^p(R^n)$, which is the Lebesgue spaces.

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [3][4][10][14]).

In this paper, we will study some multiplier operator as following (see [8][12] [20–22]).

A bounded measurable function k defined on $R^n\setminus\{0\}$ is called a multiplier. The multiplier operator T associated with k is defined by

$$T(f)(x)=k(x)\hat{f}(x),\ \text{for } f\in S(R^n),$$

where $\hat{f}$ denotes the Fourier transform of f and $S(R^n)$ is the Schwartz test function class. Now, we recall the definition of the class $M(s,l)$. Denote by $|x|\sim t$ the fact that the value of x lies in the annulus $\{x\in R^n : at<|x|<bt\}$, where $0<a\le 1<b<\infty$ are values specified in each instance.

DEFINITION 2. Let $l\ge 0$ be a real number and $1\le s\le 2$. we say that the multiplier k satisfies the condition $M(s,l)$, if

$$\left(\int_{|\xi|\sim R}|D^\alpha k(\xi)|^s d\xi\right)^{\frac{1}{s}}<CR^{n/s-|\alpha|}$$

for all $R>0$ and multi-indices α with $|\alpha|\le l$, when l is a positive integer, and, in addition, if

$$\left(\int_{|\xi|\sim R}|D^\alpha k(\xi)-D^\alpha k(\xi-z)|^s d\xi\right)^{\frac{1}{s}}\le C(\frac{|z|}{R})^\gamma R^{\frac{n}{s}-|\alpha|}$$

for all $|z|<R/2$ and all multi-indices α with $|\alpha|=[l]$, the integer part of l,i.e., $[l]$ is the greatest integer less than or equal to l, and $l=[l]+\gamma$ when l is not an integer.

DEFINITION 3. For a real number $\tilde{l}\ge 0$ and $1\le\tilde{s}<\infty$, we say that K verifies the condition $\tilde{M}(\tilde{s},\tilde{l})$, and write $K\in\tilde{M}(\tilde{s},\tilde{l})$, if

$$\left(\int_{|x|\sim R}|D^{\tilde{\alpha}}K(x)|^{\tilde{s}}dx\right)^{\frac{1}{\tilde{s}}}\le CR^{n/\tilde{s}-n-|\tilde{\alpha}|},\quad R>0$$

for all multi-indices $|\tilde{\alpha}| \le \tilde{l}$ and, in addition, if

$$\begin{aligned}&\left(\int_{|x|\sim R} |D^{\tilde{\alpha}}K(x) - D^{\tilde{\alpha}}K(x-z)|^{\tilde{s}}dx\right)^{\frac{1}{\tilde{s}}}\\ &\le \quad C\left(\frac{|z|}{R}\right)^{v} R^{\frac{n}{\tilde{s}}-n-u}, \quad \text{if } 0<v<1,\end{aligned}$$

$$\begin{aligned}&\left(\int_{|x|\sim R} |D^{\tilde{\alpha}}K(x) - D^{\tilde{\alpha}}K(x-z)|^{\tilde{s}}dx\right)^{\frac{1}{\tilde{s}}}\\ &\le \quad C\left(\frac{|z|}{R}\right)\left(\log\frac{R}{|z|}\right) R^{\frac{n}{\tilde{s}}-n-u}, \quad \text{if } v=1,\end{aligned}$$

for all $|z| < \frac{R}{2}, R > 0$, and all multi-indices $\tilde{\alpha}$ with$|\tilde{\alpha}| = u$, where u denotes the largest integer strictly less than $\tilde{l}$ with $\tilde{l} = u + v$.

Denote $D(R^n) = \{\phi \in S(R^n) : \text{supp}(\phi) \text{ is compact}\}$ and $\hat{D}_0(R^n) = \{\phi \in S(R^n) : \hat{\phi} \in D(R^n)$ and $\hat{\phi}$ vanishes in a neighbourhood of the origin$\}$. The following boundedness property of T on $L^p(R^n)$ is proved by Strömberg and Torkinsky (see [11-14]).

Lemma 1 (see [8]). *Let $k \in M(s,l), 1 \le s \le 2$, and $l > \frac{n}{s}$. Then the associated mapping T, defined a priori for $f \in \hat{D}_0(R^n)$, $T(f)(x) = (f * K)(x)$, extends to a bounded mapping from $L^p(R^n)$ into itself for $1 < p < \infty$ and $K(x) = \check{k}(x)$.*

Lemma 2 (see [8]). *Suppose $k \in M(s,l)$, $1 \le s \le 2$. Given $1 \le \tilde{s} < \infty$, let $r \ge 1$ be such that $\frac{1}{r} = \max\{\frac{1}{s}, 1 - \frac{1}{\tilde{s}}\}$. Then $K \in \widetilde{M}(\tilde{s}, \tilde{l})$, where $\tilde{l} = l - \frac{n}{r}$.*

Lemma 3 (see [8]). *Let $1 \le s < \infty$, suppose that l is a positive real number with $l > n/r$, $1/r = \max\{1/s, 1-1/\tilde{s}\}$, and $k \in M(s,l)$. Then there is a positive constant a, such that*

$$\left(\int_{Q_k} |K(x-z) - K(x_Q - z)|^{\tilde{s}}dz\right)^{1/\tilde{s}} \le C2^{-ka}(2^k h)^{-n/\tilde{s}'}.$$

Now we can define the Toeplitz type operator associated to the multiplier operator as following.

Let b be a locally integrable function on R^n and T be the multiplier operator. By Lemma 1, $T(f)(x) = (K*f)(x)$ for $K(x) = \check{k}(x)$. The Toeplitz type operator associated to T is defined by

$$T_b = \sum_{k=1}^{m}(T^{k,1}M_b I_\alpha T^{k,2} + T^{k,3} I_\alpha M_b T^{k,4}),$$

where $T^{k,1}$ are the multiplier operator T or $\pm I$(the identity operator), $T^{k,2}$ and $T^{k,4}$ are the linear operators, $T^{k,3} = \pm I$, $k = 1, ..., m$, $M_b(f) = bf$ and I_α is the fractional integral operator($0 < \alpha < n$)(see [1]).

Note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz type operator T_b. The Toeplitz type operator T_b is the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [18]). The main purpose of this paper is to prove the sharp maximal inequalities for the Toeplitz type operator T_b. As the application, we obtain the L^p-norm inequality, Morrey and Triebel-Lizorkin spaces boundedness for the Toeplitz type operator T_b.

2. Theorems and Lemmas

We shall prove the following theorems.

Theorem 1. *Let T be the multiplier operator, $0 < \beta < \min(1, a)$, $1 < s < \infty$ and $b \in Lip_\beta(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M^\#(T_b(f))(\tilde{x}) \le C||b||_{Lip_\beta} \sum_{k=1}^m (M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Theorem 2. *Let T be the multiplier operator, $0 < \beta < 1$, $1 < s < \infty$ and $b \in Lip_\beta(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$\begin{aligned} &\sup_{Q \ni \tilde{x}} \inf_{c \in R} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - c| \, dx \\ \le\ & C||b||_{Lip_\beta} \sum_{k=1}^m (M_s(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\alpha,s}(T^{k,4}(f))(\tilde{x})). \end{aligned}$$

Theorem 3. *Let T be the multiplier operator, $1 < s < \infty$ and $b \in BMO(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M^\#(T_b(f))(\tilde{x}) \le C||b||_{BMO} \sum_{k=1}^m (M_s(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Theorem 4. *Let T be the multiplier operator, $0 < \beta < \min(1, a)$, $1 < p < n/(\alpha + \beta)$, $1/q = 1/p - (\alpha + \beta)/n$ and $b \in Lip_\beta(R^n)$. If $T_1(g) = 0$ for any $g \in L^u(R^n)(1 < u < \infty)$ and $T^{k,2}$ and $T^{k,4}$ are the bounded operators on $L^p(R^n)$ for $1 < p < \infty$, $k = 1, ..., m$, then T_b is bounded from $L^p(R^n)$ to $L^q(R^n)$.*

THEOREM 5. *Let T be the multiplier operator, $0<\beta<1$, $1<p<n/(\alpha+\beta)$, $1/q=1/p-(\alpha+\beta)/n$, $0<D<2^n$ and $b\in Lip_\beta(R^n)$. If $T_1(g)=0$ for any $g\in L^u(R^n)(1<u<\infty)$ and $T^{k,2}$ and $T^{k,4}$ are the bounded operators on $L^{p,\varphi}(R^n)$ for $1<p<\infty$, $k=1,...,m$, then T_b is bounded from $L^{p,\alpha+\beta,\varphi}(R^n)$ to $L^{q,\varphi}(R^n)$.*

THEOREM 6. *Let T be the multiplier operator, $0<\beta<\min(1,a)$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$ and $b\in Lip_\beta(R^n)$. If $T_1(g)=0$ for any $g\in L^u(R^n)(1<u<\infty)$ and $T^{k,2}$ and $T^{k,4}$ are the bounded operators on $L^p(R^n)$ for $1<p<\infty$, $k=1,...,m$, then T_b is bounded from $L^p(R^n)$ to $\dot{F}_q^{\beta,\infty}(R^n)$.*

THEOREM 7. *Let T be the multiplier operator, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$ and $b\in BMO(R^n)$. If $T_1(g)=0$ for any $g\in L^u(R^n)(1<u<\infty)$ and $T^{k,2}$ and $T^{k,4}$ are the bounded operators on $L^p(R^n)$ for $1<p<\infty$, $k=1,...,m$, then T_b is bounded from $L^p(R^n)$ to $L^q(R^n)$.*

THEOREM 8. *Let T be the multiplier operator, $0<D<2^n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$ and $b\in BMO(R^n)$. If $T_1(g)=0$ for any $g\in L^u(R^n)(1<u<\infty)$ and $T^{k,2}$ and $T^{k,4}$ are the bounded operators on $L^{p,\varphi}(R^n)$ for $1<p<\infty$, $k=1,...,m$, then T_b is bounded from $L^{p,\alpha,\varphi}(R^n)$ to $L^{q,\varphi}(R^n)$.*

COROLLARY 1. *Let $[b,T](f)=bT(f)-T(bf)$ be the commutator generated by the multiplier operator T and b. Then Theorems 1-8 hold for $[b,T]$.*

To prove the theorems, we need the following lemmas.

LEMMA 4 (see [15]). *For $0<\beta<1$ and $1<p<\infty$, we have*

$$\begin{aligned}||f||_{\dot{F}_p^{\beta,\infty}} &\approx \left|\left|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q|f(x)-f_Q|dx\right|\right|_{L^p}\\ &\approx \left|\left|\sup_{Q\ni\cdot}\inf_c\frac{1}{|Q|^{1+\beta/n}}\int_Q|f(x)-c|dx\right|\right|_{L^p}.\end{aligned}$$

LEMMA 5 (see [5]). *Let $0<p<\infty$ and $w\in\cup_{1\le r<\infty}A_r$. Then, for any smooth function f for which the left-hand side is finite,*

$$\int_{R^n}M(f)(x)^p w(x)dx\le C\int_{R^n}M^{\#}(f)(x)^p w(x)dx.$$

LEMMA 6 (see [1][5]). *Suppose that $0<\alpha<n$, $1\le s<p<n/\alpha$ and $1/q=1/p-\alpha/n$. Then*

$$||I_\alpha(f)||_{L^q}\le C||f||_{L^p}$$

and

$$||M_{\alpha,s}(f)||_{L^q}\le C||f||_{L^p}.$$

LEMMA 7. *Let $1 < p < \infty$, $0 < D < 2^n$. Then, for any smooth function f for which the left-hand side is finite,*

$$||M(f)||_{L^{p,\varphi}} \leq C||M^{\#}(f)||_{L^{p,\varphi}}.$$

PROOF. For any cube $Q = Q(x_0, d)$ in R^n, we know $M(\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [5]. Noticing that $M(\chi_Q) \leq 1$ and $M(\chi_Q)(x) \leq d^n/(|x-x_0|-d)^n$ if $x \in Q^c$, by Lemma 3, we have, for $f \in L^{p,\varphi}(R^n)$,

$$\begin{aligned}
\int_Q M(f)(x)^p dx &= \int_{R^n} M(f)(x)^p \chi_Q(x) dx \\
&\leq \int_{R^n} M(f)(x)^p M(\chi_Q)(x) dx \leq C \int_{R^n} M^{\#}(f)(x)|^p M(\chi_Q)(x) dx \\
&= C\left(\int_Q M^{\#}(f)(x)^p M(\chi_Q)(x) dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^{\#}(f)(x)^p M(\chi_Q)(x) dx\right) \\
&\leq C\left(\int_Q M^{\#}(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^{\#}(f)(x)^p \frac{|Q|}{|2^{k+1}Q|} dx\right) \\
&\leq C\left(\int_Q M^{\#}(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q} M^{\#}(f)(x)^p 2^{-kn} dy\right) \\
&\leq C||M^{\#}(f)||^p_{L^{p,\varphi}} \sum_{k=0}^{\infty} 2^{-kn} \varphi(2^{k+1}d) \\
&\leq C||M^{\#}(f)||^p_{L^{p,\varphi}} \sum_{k=0}^{\infty} (2^{-n}D)^k \varphi(d) \\
&\leq C||M^{\#}(f)||^p_{L^{p,\varphi}} \varphi(d),
\end{aligned}$$

thus

$$\left(\frac{1}{\varphi(d)} \int_Q M(f)(x)^p dx\right)^{1/p} \leq C\left(\frac{1}{\varphi(d)} \int_Q M^{\#}(f)(x)^p dx\right)^{1/p}$$

and

$$||M(f)||_{L^{p,\varphi}} \leq C||M^{\#}(f)||_{L^{p,\varphi}}.$$

This finishes the proof. □

LEMMA 8. *Let $0 < \alpha < n$, $0 < D < 2^n$, $1 \leq s < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Then*

$$||I_\alpha(f)||_{L^{q,\varphi}} \leq C||f||_{L^{p,\alpha,\varphi}}$$

and

$$||M_{\alpha,s}(f)||_{L^{r,\varphi}} \leq C||f||_{L^{p,\alpha,\varphi}}.$$

The proof of the Lemma is similar to that of Lemma 7 by Lemma 6, we omit the details.

3. Proofs of Theorems

PROOF OF THEOREM 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |T_b(f)(x) - C_0|\,dx \\ \leq\ & C||b||_{Lip_\beta}\sum_{k=1}^m (M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x})).\end{aligned}$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, ..., m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We write, by $T_1(g) = 0$,

$$\begin{aligned}T_b(f)(x) &= \sum_{k=1}^m T^{k,1}M_bI_\alpha T^{k,2}(f)(x) + \sum_{k=1}^m T^{k,3}I_\alpha M_bT^{k,4}(f)(x) \\ &= A_b(x) + B_b(x) = A_{b-b_{2Q}}(x) + B_{b-b_{2Q}}(x),\end{aligned}$$

where

$$\begin{aligned}A_{b-b_{2Q}}(x) &= \sum_{k=1}^m T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(x) \\ &\quad + \sum_{k=1}^m T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x) \\ &= A_1(x) + A_2(x)\end{aligned}$$

and

$$\begin{aligned}B_{b-b_{2Q}}(x) &= \sum_{k=1}^m T^{k,3}I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)(x) \\ &\quad + \sum_{k=1}^m T^{k,3}I_\alpha M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,4}(f)(x) \\ &= B_1(x) + B_2(x).\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |T_b(f)(x)-A_2(x_0)-B_2(x_0)|\,dx\\
&\le \frac{1}{|Q|}\int_Q |A_1(x)|dx+\frac{1}{|Q|}\int_Q |A_2(x)-A_2(x_0)|dx\\
&\quad +\frac{1}{|Q|}\int_Q |B_1(x)|dx+\frac{1}{|Q|}\int_Q |B_2(x)-B_2(x_0)|dx\\
&= I_1+I_2+I_3+I_4.
\end{aligned}$$

For I_1, by Hölder's inequality and Lemma 1, we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(x)|dx\\
&\le \left(\frac{1}{|Q|}\int_{R^n} |T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s}\\
&\le C|Q|^{-1/s}\left(\int_{R^n} |M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s}\\
&\le C|Q|^{-1/s}\left(\int_{2Q} (|b(x)-b_{2Q}||I_\alpha T^{k,2}(f)(x)|)^s dx\right)^{1/s}\\
&\le C|Q|^{-1/s}||b||_{Lip_\beta}|2Q|^{\beta/n}|2Q|^{1/s-\beta/n}\\
&\quad\times\left(\frac{1}{|2Q|^{1-s\beta/n}}\int_{2Q} |I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s}\\
&\le C||b||_{Lip_\beta}M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_1 &\le \sum_{k=1}^m \frac{1}{|Q|}\int_{R^n} |T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(x)|dx\\
&\le C||b||_{Lip_\beta}\sum_{k=1}^m M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

For I_2, we get, for $x \in Q$,

$$\begin{aligned}
&|T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x) - T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x_0)| \\
&\leq \int_{(2Q)^c} |b(y) - b_{2Q}||K(x-y) - K(x_0-y)||I_\alpha T^{k,2}(f)(y)|dy \\
&\leq C\sum_{j=1}^\infty \int_{2^j d \leq |y-x_0| < 2^{j+1}d} ||b||_{Lip_\beta}(2^{j+1}d)^\beta |K(x-y) - K(x_0-y)| \\
&\quad \times |I_\alpha T^{k,2}(f)(y)|dy \\
&\leq C||b||_{Lip_\beta}\sum_{j=1}^\infty (2^j d)^\beta \left(\int_{|y-x_0|<2^{j+1}d} |I_\alpha T^{k,2}(f)(y)|^s dy\right)^{1/s} \\
&\quad \times \left(\int_{2^j d \leq |y-x_0| < 2^{j+1}d} |K(x-y) - K(x_0-y)|^{s'} dy\right)^{1/s'} \\
&\leq C||b||_{Lip_\beta}\sum_{j=1}^\infty 2^{j(\beta-a)} \left(\frac{1}{|2^{j+1}Q|^{1-s\beta/n}}\int_{2^{j+1}Q} |I_\alpha T^{k,2}(f)(y)|^s dy\right)^{1/s} \\
&\leq C||b||_{Lip_\beta} M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_2 &\leq \frac{1}{|Q|}\int_Q \sum_{k=1}^m |T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x) \\
&\quad - T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x_0)|dx \\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^m M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

Similarly, by Lemma 6, for $1/r = 1/s - \alpha/n$,

$$\begin{aligned}
I_3 &\le \sum_{k=1}^{m} \left(\frac{1}{|Q|}\int_{R^n} |I_\alpha M_{(b-b_{2Q})\chi_{2Q}} T^{k,4}(f)(x)|^r dx\right)^{1/r} \\
&\le C\sum_{k=1}^{m} |Q|^{-1/r} \left(\int_{2Q} (|b(x)-b_{2Q}||T^{k,4}(f)(x)|)^s dx\right)^{1/s} \\
&\le C||b||_{Lip_\beta} \sum_{k=1}^{m} |Q|^{-1/r}|2Q|^{\beta/n}|2Q|^{1/s-(\beta+\alpha)/n} \\
&\quad \times \left(\frac{1}{|2Q|^{1-s(\beta+\alpha)/n}}\int_{2Q} |T^{k,4}(f)(x)|^s dx\right)^{1/s} \\
&\le C||b||_{Lip_\beta} \sum_{k=1}^{m} M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
I_4 &\le \sum_{k=1}^{m} \frac{1}{|Q|}\int_Q \int_{(2Q)^c} |b(y)-b_{2Q}| \left|\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_0-y|^{n-\alpha}}\right| \\
&\quad \times |T^{k,4}(f)(y)|dydx \\
&\le C\sum_{k=1}^{m}\sum_{j=1}^{\infty} ||b||_{Lip_\beta}|2^{j+1}Q|^{\beta/n} \int_{2^j d\le|y-x_0|<2^{j+1}d} \frac{d}{|x_0-y|^{n-\alpha+1}} \\
&\quad \times |T^{k,4}(f)(y)|dy \\
&\le C||b||_{Lip_\beta} \sum_{k=1}^{m}\sum_{j=1}^{\infty} (2^j d)^\beta d (2^j d)^{-n+\alpha-1} (2^j d)^{n(1-1/s)} (2^j d)^{n/s-\beta-\alpha} \\
&\quad \times \left(\frac{1}{|2^{j+1}Q|^{1-s(\beta+\alpha)/n}} \int_{2^{j+1}Q} |T^{k,4}(f)(y)|^s dy\right)^{1/s} \\
&\le C||b||_{Lip_\beta} \sum_{k=1}^{m} M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x}) \sum_{j=1}^{\infty} 2^{-j} \\
&\le C||b||_{Lip_\beta} \sum_{k=1}^{m} M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x}).
\end{aligned}$$

These complete the proof of Theorem 1. □

Proof of Theorem 2. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q |T_b(f)(x)-C_0|\,dx\\
&\quad\leq\; C||b||_{Lip_\beta}\sum_{k=1}^m (M_s(I_\alpha T^{k,2}(f))(\tilde{x})+M_{\alpha,s}(T^{k,4}(f))(\tilde{x})).
\end{aligned}$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k=1,...,m)$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Similar to the proof of Theorem 1, we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q |T_b(f)(x)-A_2(x_0)-B_2(x_0)|\,dx\\
\leq\;& \frac{1}{|Q|^{1+\beta/n}}\int_Q |A_1(x)|dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q |A_2(x)-A_2(x_0)|dx\\
&+\frac{1}{|Q|^{1+\beta/n}}\int_Q |B_1(x)|dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q |B_2(x)-B_2(x_0)|dx\\
=\;& J_1+J_2+J_3+J_4.
\end{aligned}$$

By using the same argument as in the proof of Theorem 1, we get, for $1/r = 1/s-\alpha/n$,

$$\begin{aligned}
J_1 \;\leq\;& |Q|^{-\beta/n}\sum_{k=1}^m\left(\frac{1}{|Q|}\int_{R^n}|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s}\\
\leq\;& C|Q|^{-\beta/n}\sum_{k=1}^m |Q|^{-1/s}\left(\int_{2Q}(|b(x)-b_{2Q}||I_\alpha T^{k,2}(f)(x)|)^s dx\right)^{1/s}\\
\leq\;& C|Q|^{-\beta/n}\sum_{k=1}^m |Q|^{-1/s}||b||_{Lip_\beta}|2Q|^{\beta/n}|Q|^{1/s}\\
&\times\left(\frac{1}{|2Q|}\int_{2Q}|I_\alpha T^{k,2}(f)(x)|^s dx\right)^{1/s}\\
\leq\;& C||b||_{Lip_\beta}\sum_{k=1}^m M_s(I_\alpha T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
J_2 &\le |Q|^{-\beta/n}\sum_{k=1}^{m}\frac{1}{|Q|}\int_Q\sum_{j=1}^{\infty}\int_{2^jd\le|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\\
&\quad\times|K(x-y)-K(x_0-y)||I_\alpha T^{k,2}(f)(y)|dydx\\
&\le C||b||_{Lip_\beta}|Q|^{-\beta/n}\sum_{k=1}^{m}\frac{1}{|Q|}\int_Q\sum_{j=1}^{\infty}|2^{j+1}Q|^{\beta/n}\\
&\quad\times\left(\int_{|y-x_0|<2^{j+1}d}|I_\alpha T^{k,2}(f)(y)|^s dy\right)^{1/s}\\
&\quad\times\left(\int_{2^jd\le|y-x_0|<2^{j+1}d}|K(x-y)-K(x_0-y)|^{s'}dy\right)^{1/s'}\\
&\le C||b||_{Lip_\beta}\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{-j\beta}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|I_\alpha T^{k,2}(f)(y)|^s dy\right)^{1/s}\\
&\le C||b||_{Lip_\beta}\sum_{k=1}^{m}M_s(I_\alpha T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
J_3 &\le |Q|^{-\beta/n}\sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_{R^n}|I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)(x)|^r dx\right)^{1/r}\\
&\le C|Q|^{-\beta/n-1/r}\sum_{k=1}^{m}\left(\int_{2Q}(|b(x)-b_{2Q}||T^{k,4}(f)(x)|)^s dx\right)^{1/s}\\
&\le C||b||_{Lip_\beta}\sum_{k=1}^{m}|Q|^{-\beta/n-1/r}|2Q|^{\beta/n}|Q|^{1/s-\alpha/n}\\
&\quad\times\left(\frac{1}{|2Q|^{1-s\alpha/n}}\int_{2Q}|T^{k,4}(f)(x)|^s dx\right)^{1/s}\\
&\le C||b||_{Lip_\beta}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,4}(f))(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
J_4 &\leq |Q|^{-\beta/n-1}\sum_{k=1}^{m}\int_Q\int_{(2Q)^c}|b(y)-b_{2Q}|\left|\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x_0-y|^{n-\alpha}}\right| \\
&\quad \times|T^{k,4}(f)(y)|dydx \\
&\leq C|Q|^{-\beta/n}\sum_{k=1}^{m}\sum_{j=1}^{\infty}||b||_{Lip_\beta}|2^{j+1}Q|^{\beta/n}\int_{2^jd\leq|y-x_0|<2^{j+1}d}\frac{d}{|x_0-y|^{n-\alpha+1}} \\
&\quad \times|T^{k,4}(f)(y)|dy \\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^{m}\sum_{j=1}^{\infty}d^{-\beta}(2^jd)^{\beta}d(2^jd)^{-n+\alpha-1}(2^jd)^{n(1-1/s)}(2^jd)^{n/s-\alpha} \\
&\quad \times\left(\frac{1}{|2^{j+1}Q|^{1-s\alpha/n}}\int_{2^{j+1}Q}|T^{k,4}(f)(y)|^sdy\right)^{1/s} \\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,4}(f))(\tilde{x})\sum_{j=1}^{\infty}2^{j(\beta-1)} \\
&\leq C||b||_{Lip_\beta}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,4}(f))(\tilde{x}).
\end{aligned}$$

These complete the proof of Theorem 2. □

Proof of Theorem 3. It suffices to prove for $f\in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q|T_b(f)(x)-C_0|\,dx \\
&\quad\leq C||b||_{BMO}\sum_{k=1}^{m}(M_s(I_\alpha T^{k,2}(f))(\tilde{x})+M_{\alpha,s}(T^{k,4}(f))(\tilde{x})).
\end{aligned}$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k=1,...,m)$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Similar to the proof of Theorem 1, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q|T_b(f)(x)-A_2(x_0)-B_2(x_0)|\,dx \\
&\quad\leq \frac{1}{|Q|}\int_Q|A_1(x)|dx+\frac{1}{|Q|}\int_Q|A_2(x)-A_2(x_0)|dx \\
&\qquad+\frac{1}{|Q|}\int_Q|B_1(x)|dx+\frac{1}{|Q|}\int_Q|B_2(x)-B_2(x_0)|dx \\
&\quad= L_1+L_2+L_3+L_4.
\end{aligned}$$

By using the same argument as in the proof of Theorem 1, we get, for $1 < r_1 < s$, $1 < r_2, r_3 < \infty$ with $1/r_2 + 1/r_3 + 1/s = 1$, $1 < p < \min(s, n/\alpha)$ with $1/r_4 = 1/p - \alpha/n$,

$$\begin{aligned}
L_1 &\leq \sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_{R^n}|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(x)|^{r_1}dx\right)^{1/r_1}\\
&\leq C\sum_{k=1}^{m}|Q|^{-1/r}\left(\int_{2Q}(|b(x)-b_{2Q}||I_\alpha T^{k,2}(f)(x)|)^{r_1}dx\right)^{1/r_1}\\
&\leq C\sum_{k=1}^{m}\left(\frac{1}{|2Q|}\int_{2Q}|I_\alpha T^{k,2}(f)(x)|^{s}dx\right)^{1/s}\\
&\quad\times\left(\frac{1}{|2Q|}\int_{2Q}|b(x)-b_{2Q}|^{sr_1/(s-r_1)}dx\right)^{(s-r_1)/sr_1}\\
&\leq C||b||_{BMO}\sum_{k=1}^{m}M_s(I_\alpha T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
L_2 &\leq \sum_{k=1}^{m}\frac{1}{|Q|}\int_{Q}\sum_{j=1}^{\infty}\int_{2^jd\leq|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}||K(x-y)-K(x_0-y)|\\
&\quad\times|I_\alpha T^{k,2}(f)(y)|dydx\\
&\leq \sum_{k=1}^{m}\frac{C}{|Q|}\int_{Q}\sum_{j=1}^{\infty}\left(\int_{2^{j+1}Q}|b(y)-b_{2Q}|^{r_2}dy\right)^{1/r_2}\\
&\quad\times\left(\int_{2^{j+1}Q}|I_\alpha T^{k,2}(f)(y)|^{s}dy\right)^{1/s}\\
&\quad\times\left(\int_{2^jd\leq|y-x_0|<2^{j+1}d}|K(x-y)-K(x_0-y)|^{r_3}dy\right)^{1/r_3}dx\\
&\leq C||b||_{BMO}\sum_{k=1}^{m}\sum_{j=1}^{\infty}j2^{-j}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|I_\alpha T^{k,2}(f)(y)|^{s}dy\right)^{1/s}\\
&\leq C||b||_{BMO}\sum_{k=1}^{m}M_s(I_\alpha T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

$$
\begin{aligned}
L_3 &\leq \sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_{R^n}|I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)(x)|^{r_4}dx\right)^{1/r_4}\\
&\leq C|Q|^{-1/r_2}\sum_{k=1}^{m}\left(\int_{2Q}(|b(x)-b_{2Q}||T^{k,4}(f)(x)|)^p dx\right)^{1/p}\\
&\leq C\sum_{k=1}^{m}\left(\frac{1}{|2Q|}\int_{2Q}|b(x)-b_{2Q}|^{ps/(s-p)}dx\right)^{(s-p)/ps}\\
&\quad\times\left(\frac{1}{|2Q|^{1-s\alpha/n}}\int_{2Q}|T^{k,4}(f)(x)|^s dx\right)^{1/s}\\
&\leq C||b||_{BMO}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,4}(f))(\tilde{x}),
\end{aligned}
$$

$$
\begin{aligned}
L_4 &\leq |Q|^{-1}\sum_{k=1}^{m}\int_Q\int_{(2Q)^c}|b(y)-b_{2Q}|\left|\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x_0-y|^{n-\alpha}}\right|\\
&\quad\times|T^{k,4}(f)(y)|dydx\\
&\leq C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\int_{2^jd\leq|y-x_0|<2^{j+1}d}|b(y)-b_{2Q}|\frac{d}{|x_0-y|^{n-\alpha+1}}|T^{k,4}(f)(y)|dy\\
&\leq C\sum_{k=1}^{m}\sum_{j=1}^{\infty}d(2^jd)^{-n+\alpha-1}(2^jd)^{n(1-1/s)}(2^jd)^{n/s-\alpha}\\
&\quad\times\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}|b(y)-b_{2Q}|^{s'}dy\right)^{1/s'}\\
&\quad\times\left(\frac{1}{|2^{j+1}Q|^{1-s\alpha/n}}\int_{2^{j+1}Q}|T^{k,4}(f)(y)|^s dy\right)^{1/s}\\
&\leq C||b||_{BMO}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,4}(f))(\tilde{x})\sum_{j=1}^{\infty}j2^{-j}\\
&\leq C||b||_{BMO}\sum_{k=1}^{m}M_{\alpha,s}(T^{k,4}(f))(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 3. □

PROOF OF THEOREM 4. Choose $1 < s < p$ in Theorem 1 and set $1/r = 1/p - \alpha/n$. We have, by Lemmas 5 and 6,

$$\begin{aligned}
&||T_b(f)||_{L^q} \leq \|M(T_b(f))\|_{L^q} \leq C\|M^{\#}(T_b(f))\|_{L^q} \\
&\leq \quad C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|M_{\beta,s}(I_\alpha T^{k,2}(f))\|_{L^q} + \|M_{\beta+\alpha,s}(T^{k,4}(f))\|_{L^q}) \\
&\leq \quad C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|I_\alpha T^{k,2}(f)\|_{L^r} + \|T^{k,4}(f)\|_{L^p}) \\
&\leq \quad C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|T^{k,2}(f)\|_{L^p} + \|f\|_{L^p}) \\
&\leq \quad C||b||_{Lip_\beta} \|f\|_{L^p}.
\end{aligned}$$

This completes the proof of the theorem. □

PROOF OF THEOREM 5. Choose $1 < s < p$ in Theorem 1 and set $1/r = 1/p - \alpha/n$. We have, by Lemmas 7 and 8,

$$\begin{aligned}
&||T_b(f)||_{L^{q,\varphi}} \leq \|M(T_b(f))\|_{L^{q,\varphi}} \leq C\|M^{\#}(T_b(f))\|_{L^{q,\varphi}} \\
&\leq \quad C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|M_{\beta,s}(I_\alpha T^{k,2}(f))\|_{L^{q,\varphi}} + \|M_{\beta+\alpha,s}(T^{k,4}(f))\|_{L^{q,\varphi}}) \\
&\leq \quad C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|I_\alpha T^{k,2}(f)\|_{L^{r,\beta,\varphi}} + \|T^{k,4}(f)\|_{L^{p,\alpha+\beta,\varphi}}) \\
&\leq \quad C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|T^{k,2}(f)\|_{L^{p,\alpha+\beta,\varphi}} + \|f\|_{L^{p,\alpha+\beta,\varphi}}) \\
&\leq \quad C||b||_{Lip_\beta} \|f\|_{L^{p,\alpha+\beta,\varphi}}.
\end{aligned}$$

This completes the proof of the theorem. □

PROOF OF THEOREM 6. Choose $1 < s < p$ in Theorem 2. We have, by Lemmas 4 and 6,

$$||T_b(f)||_{\dot{F}_q^{\beta,\infty}} \le C \left|\left| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T_b(f)(x) - C_0| \, dx \right|\right|_{L^q}$$

$$\le \quad C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|M_s(I_\alpha T^{k,2}(f))\|_{L^q} + \|M_{\alpha,s}(T^{k,4}(f))\|_{L^q})$$

$$\le \quad C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|I_\alpha T^{k,2}(f)\|_{L^q} + \|T^{k,4}(f)\|_{L^p})$$

$$\le \quad C||b||_{Lip_\beta} \sum_{k=1}^{m} (\|T^{k,2}(f)\|_{L^p} + \|f\|_{L^p})$$

$$\le \quad C||b||_{Lip_\beta} ||f||_{L^p}.$$

This completes the proof of the theorem. □

PROOF OF THEOREM 7. Choose $1 < s < p$ in Theorem 3, we have, by Lemmas 5 and 6,

$$||T_b(f)||_{L^q} \le \|M(T_b(f))\|_{L^q} \le C\|M^{\#}(T_b(f))\|_{L^q}$$

$$\le \quad C||b||_{BMO} \sum_{k=1}^{m} (\|M_s(I_\alpha T^{k,2}(f))\|_{L^q} + \|M_{\alpha,s}(T^{k,4}(f))\|_{L^q})$$

$$\le \quad C||b||_{BMO} \sum_{k=1}^{m} (\|I_\alpha T^{k,2}(f)\|_{L^q} + \|T^{k,4}(f)\|_{L^p})$$

$$\le \quad C||b||_{BMO} \sum_{k=1}^{m} (\|T^{k,2}(f)\|_{L^p} + \|f\|_{L^p})$$

$$\le \quad C||b||_{BMO} \|f\|_{L^p}.$$

This completes the proof of the theorem. □

PROOF OF THEOREM 8. Choose $1 < s < p$ in Theorem 3, we have, by Lemmas 7 and 8,

$$\begin{aligned}
||T_b(f)||_{L^{q,\varphi}} &\le \|M(T_b(f))\|_{L^{q,\varphi}} \le C\|M^{\#}(T_b(f))\|_{L^{q,\varphi}} \\
&\le C||b||_{BMO}\sum_{k=1}^{m}(\|M_s(I_\alpha T^{k,2}(f))\|_{L^{q,\varphi}} + \|M_{\alpha,s}(T^{k,4}(f))\|_{L^{q,\varphi}}) \\
&\le C||b||_{BMO}\sum_{k=1}^{m}(\|I_\alpha T^{k,2}(f)\|_{L^{q,\varphi}} + \|T^{k,4}(f)\|_{L^{p,\alpha,\varphi}}) \\
&\le C||b||_{BMO}\sum_{k=1}^{m}(\|T^{k,2}(f)\|_{L^{p,\alpha,\varphi}} + \|f\|_{L^{p,\alpha,\varphi}}) \\
&\le C||b||_{BMO}\|f\|_{L^{p,\alpha,\varphi}}.
\end{aligned}$$

This completes the proof of the theorem. □

REFERENCES

1. S. Chanillo, A note on commutators, Indiana Univ. Math. J., **31**(1982), 7-16.
2. R. R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math., 103(1976), 611-635.
3. G. Di FaZio and M. A. Ragusa, Commutators and Morrey spaces, Boll. Un. Mat. Ital., 5-A(7)(1991), 323-332.
4. G. Di Fazio and M. A. Ragusa, Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients, J. Func. Anal., 112(1993), 241-256.
5. J. Garcia-Cuerva and J. L. Rubio de Francia, Weighted norm inequalities and related topics, North-Holland Math.,16, Amsterdam, 1985.
6. S. Janson, Mean oscillation and commutators of singular integral operators, Ark. Math., 16(1978), 263-270.
7. S. Krantz and S. Li, Boundedness and compactness of integral operators on spaces of homogeneous type and applications, J. of Math. Anal. Appl., 258(2001), 629-641.
8. D. S. Kurtz and R. L. Wheeden, Results on weighted norm inequalities for multiplies, Trans. Amer. Math. Soc., 255(1979), 343-362.
9. Y. Lin and S. Z. Lu, Toeplitz type operators associated to strongly singular integral operator, Sci. in China(ser.A), 36(2006), 615-630.
10. L. Z. Liu, Interior estimates in Morrey spaces for solutions of elliptic equations and weighted boundedness for commutators of singular integral operators, Acta Math. Scientia, 25(B)(1)(2005), 89-94.
11. S. Z. Lu and H. X. Mo, Toeplitz type operators on Lebesgue spaces, Acta Math. Scientia, 29(B)(1)(2009), 140-150.
12. S. Z. Lu and D. C. Yang, Multiplier theorem for Herz type Hardy spaces, Proc. Amer. Math. Soc., 126(1998), 3337-3346.
13. B. Muckenhoupt, R. L. Wheeden and W. S. Young, Sufficient conditions for L^p multipliers with general weights, Trans. Amer. Math. Soc., 300(2)(1987), 463-502.
14. T. Mizuhara, Boundedness of some classical operators on generalized Morrey spaces, in "Harmonic Analysis", Proceedings of a conference held in Sendai, Japan, 1990, 183-189.
15. M. Paluszynski, Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss, Indiana Univ. Math. J., 44(1995), 1-17.
16. J. Peetre, On convolution operators leaving $L^{p,\lambda}$-spaces invariant, Ann. Mat. Pura. Appl., 72(1966), 295-304.

17. J. Peetre, On the theory of $L^{p,\lambda}$-spaces, J. Func. Anal., 4(1969), 71-87.
18. C. Pérez and R. Trujillo-Gonzalez, Sharp weighted estimates for multilinear commutators, J. London Math. Soc., 65(2002), 672-692.
19. E. M. Stein, Harmonic analysis: real variable methods, orthogonality and oscillatory integrals, Princeton Univ. Press, Princeton NJ, 1993.
20. Z. You, Results of commutators obtained norm inequalities, Adv. in Math.(in Chinese), 17(1)(1988), 79-84.
21. P. Zhang and J. C. Chen, The $(L^p, \dot{F}_p^{\beta,\infty})$-boundedness of commutators of multiplies, Acta Math. Sinica, 21(4)(2005), 765-772.
22. P. Zhang and J. C. Chen, Boundedness properties for commutators of multiplies, Acta Math. Sinica (Chinese Series), 49(6)(2006), 1387-1396.

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