

## LAPLACIANS FOR DERIVED GRAPHS OF REGULAR KÄHLER GRAPHS

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**ABSTRACT.** We consider  $(p, q)$ -step adjacency on a Kähler graph which is compounded from a principal graph and an auxiliary graph. We attach probabilistic weights to  $(p, q)$ -step paths so that these paths show trajectories under the influence of a magnetic field of strength  $q/p$  on this graph. We study eigenvalues of Laplacians corresponding to  $(p, q)$ -step paths on some regular Kähler graphs and give examples of pairs of regular Kähler graphs whose Laplacians for arbitrary pairs  $(p, q)$  of positive integers have the same eigenvalues.

**RÉSUMÉ.** On considère la relation de contiguïté à un pas  $(p, q)$  près sur un graphe kählerien qui se constitue d'un graphe principal et un graphe auxiliaire. On attache des poids probabilistiques aux sentiers dans un tel graphe composés de pas  $(p, q)$  de sorte que ces sentiers montrent des trajectoires sous l'influence d'un champ magnétique de force  $q/p$  sur le graphe. On étudie le spectre de l'opérateur laplacien correspondant aux sentiers aux pas  $(p, q)$  sur certains graphes kähleriens réguliers. On donne un exemple de deux graphes différents dont les opérateurs laplaciens ont le même spectre pour toute paire  $(p, q)$ .

**1. Introduction** A graph is said to be *Kähler* if its set of edges consists of principal edges and auxiliary edges. This means that a Kähler graph is a graph which is compounded from two graphs having the same set of vertices; one is a principal graph and the other is an auxiliary graph. Such graphs were introduced by the second author in [1] to give a discrete model of a Riemannian manifold admitting a magnetic field. A *magnetic field* is a closed 2-form on a Riemannian manifold and is a generalization of static magnetic fields on a Euclidean 3-space. He studied some properties of trajectories, which are motions of charged particles under the action of a magnetic field, from a classical point of view by comparing them with geodesics (see [2] and its references). On a graph we usually regard paths, which are chains of edges, as geodesics. Therefore, he intend to give a notion of trajectories for a magnetic field on some graphs.

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For ordinary graphs, their eigenvalues of adjacency operators have been studied to characterize or to show some properties of graphs (see [4], for example). For Kähler graphs the authors studied Laplacians in [10] which are generating operators of random walks having principal edges and auxiliary edges alternatively. They gave examples of isospectral pairs of Kähler graphs. Since paths where principal edges and auxiliary edges appear can alternatively be regarded as trajectories for a magnetic field of strength 1, we are interested in paths where  $p$ -principal edges and  $q$ -auxiliary edges appear alternatively and that can be regarded as trajectories for a magnetic field of strength  $q/p$ . In this paper we study  $(p, q)$ -Laplacians corresponding to such paths on some special kind of regular Kähler graphs, and give examples of pairs of finite regular Kähler graphs whose  $(p, q)$ -Laplacians have the same eigenvalues.

**2. Bicolored Paths and Derived Graphs** Let  $G = (V, E)$  be a locally finite graph, which consists of a set  $V$  of vertices and a set  $E$  of edges. We suppose that it is not directed and it does not have loops and multiple edges. Here, a loop is an edge joining a vertex and itself, and multiple edges are two or more edges joining a pair of vertices. We call this graph *Kähler* if the set  $E$  is divided into two disjoint subsets  $E^{(p)}, E^{(a)}$  satisfying the following conditions: At each vertex  $v \in V$ , there exist at least four edges emanating from  $v$ , and two of them are contained in  $E^{(p)}$  and two of them are contained in  $E^{(a)}$ . We call an edge in  $E^{(p)}$  principal and an edge in  $E^{(a)}$  auxiliary. For a Kähler graph  $G = (V, E^{(p)} \cup E^{(a)})$  we have two subgraphs  $G^{(p)} = (V, E^{(p)})$  and  $G^{(a)} = (V, E^{(a)})$ . We call them the *principal graph* and the *auxiliary graph* of  $G$ , respectively. The condition on edges of Kähler graphs shows that their principal and auxiliary graphs do not have hairs, which are edges of blind alleys. (For construction of Kähler graphs see [9].) For two vertices  $v, w \in V$  we denote  $v \sim_p w$  to mean they are adjacent to each other in  $(V, E^{(p)})$ . That is there is a principal edge joining them. Similarly, we denote by  $v \sim_a w$  if they are adjacent to each other in  $(V, E^{(a)})$ .

Given a pair of relatively prime positive integers  $(p, q)$ , we say a path  $\gamma = (v_0, \dots, v_{p+q}) \in V \times \dots \times V$  to be a  $(p, q)$ -primitive bicolored path if it satisfies the following conditions;

- (i)  $v_{i+1} \neq v_{i-1}$  for  $1 \leq i \leq p+q-1$ ,
- (ii)  $v_{i-1} \sim_p v_i$  for  $1 \leq i \leq p$ ,
- (iii)  $v_{i-1} \sim_a v_i$  for  $p+1 \leq i \leq p+q$ .

The first condition shows that this path does not have back-tracking. We say a path  $\gamma = (v_0, \dots, v_{m(p+q)})$  to be a  $(p, q)$ -bicolored path if all subpaths  $\gamma_j = (v_{(j-1)(p+q)}, \dots, v_{j(p+q)})$  ( $j = 1, \dots, m$ ) are  $(p, q)$ -primitive bicolored paths. For this path  $\gamma$  we put  $o(\gamma) = v_0$ ,  $t(\gamma) = v_{m(p+q)}$  and call them the origin and the terminus of  $\gamma$ . With the set  $E_{p,q}$  of all  $(p, q)$ -primitive bicolored paths we have a directed graph  $G_{p,q} = (V, E_{p,q})$ . We shall call this graph the  $(p, q)$ -derived graph of  $G$ . This graph may have loops and multiple edges.

We consider here the meaning of bicolored paths. We define Kähler graphs to give discrete models of Riemannian manifolds with magnetic fields. We consider

bicolored paths as models of trajectories for magnetic fields in the following way. On a Kähler graph, paths in its principal graph are considered as geodesics, which are motions of particles without any influences. Under the influence of a magnetic field motions of charged particles are bent. When the strength of a magnetic field is  $q/p$ , we consider that a  $p$ -step path  $(v_0, \dots, v_p)$  in the principal graph is bent and reaches to the terminus of a  $(p, q)$ -bicolored path  $(v_0, \dots, v_p, v_{p+1}, \dots, v_{p+q})$ . But as a graph is of 1-dimensional, we can not show the direction of the action of magnetic fields. We therefore treat positions of the terminus of trajectories probabilistically. For a vertex  $v \in V$  we set  $d_G^{(p)}(v)$  and  $d_G^{(a)}(v)$  the cardinalities of the sets  $\{w \in V \mid w \sim_p v\}$  and  $\{w \in V \mid w \sim_a v\}$ , respectively. We call them the *principal* and the *auxiliary degrees* at  $v$ , respectively. By the conditions we have  $d_G^{(p)}(v) \geq 2$  and  $d_G^{(a)}(v) \geq 2$  at an arbitrary  $v \in V$ . For a  $(p, q)$ -primitive bicolored path  $\gamma = (v_0, \dots, v_{p+q})$ , we define its probabilistic weight  $\omega(\gamma)$  by  $\omega(\gamma) = \{d_G^{(a)}(v_p) \prod_{j=p+1}^{p+q-1} \{d_G^{(a)}(v_j) - 1\}\}^{-1}$ . For a  $(p, q)$ -bicolored path  $\gamma = \gamma_1 \cdots \gamma_m$ , which is a chain of  $(p, q)$ -primitive bicolored paths  $\gamma_j$  ( $j = 1, \dots, m$ ), we set its *probabilistic weight* by  $\omega(\gamma) = \omega(\gamma_1) \times \cdots \times \omega(\gamma_m)$ . Thus when we consider the adjacency by  $(p, q)$ -primitive bicolored paths it is equivalent to the adjacency of the weighted oriented graph  $(G_{p,q}, \omega)$ .

**3. Laplacians for Kähler Graphs** For an ordinary finite graph  $G = (V, E)$  we define its adjacency operator  $\mathcal{A}_G$  and its transition operator  $\mathcal{P}_G$  acting on the space of functions on  $V$  by  $\mathcal{A}_G f(v) = \sum_{w \sim v} f(w)$  and  $\mathcal{P}_G f(v) = d_G(v)^{-1} \mathcal{A}_G f(v)$ , respectively. Here, for two vertices  $v, w \in V$  we denote as  $v \sim w$  when they are adjacent to each other, and denote by  $d_G(v)$  the degree at  $v$ , which is the cardinality of the set  $\{w \mid w \sim v\}$ . By use of the degree operator  $\mathcal{D}_G$  which is defined by  $\mathcal{D}_G f(v) = d_G(v) f(v)$ , we have the combinatorial Laplacian  $\Delta_{\mathcal{A}_G} = \mathcal{D}_G - \mathcal{A}_G$  and the transitional Laplacian  $\Delta_{\mathcal{P}_G} = I - \mathcal{P}_G$ . In this section we define such operators for Kähler graphs.

For a finite Kähler graph  $G = (V, E^{(p)} \cup E^{(a)})$ , we set  $C(V)$  the space of all functions on  $V$ . Given a pair  $(p, q)$  of relatively prime positive integers, we define two operators  $\mathcal{A}_{G;p,q} = \mathcal{A}_{p,q}$  and  $\mathcal{Q}_{G;p,q} = \mathcal{Q}_{p,q}$  acting on  $C(V)$  by

$$\mathcal{A}_{p,q} f(v) = \sum_{\gamma} \omega(\gamma) f(t(\gamma)), \quad \mathcal{Q}_{p,q} f(v) = \frac{1}{\sum_{\gamma} \omega(\gamma)} \sum_{\gamma} \omega(\gamma) f(t(\gamma)),$$

respectively, where in each summation  $\gamma$  runs over the set of all  $(p, q)$ -primitive bicolored paths with  $o(\gamma) = v$ . We call them the  $(p, q)$ -adjacency operator and the  $(p, q)$ -probabilistic transition operator, respectively. We should note that they are not symmetric in general. Also, we note that the  $(p, q)$ -adjacency operator is not the adjacency operator of the derived directed graph  $G_{p,q}$  because we treat bicolored paths probabilistically on the auxiliary graph. For a positive integer  $p$  and a vertex  $v \in V$ , we denote by  $d_{G;p}^{(p)}(v)$  the cardinality of the set of all  $p$ -step paths without back-tracking in the principal graph  $G^{(p)} = (V, E^{(p)})$  which emanate from  $v \in V$ . We define an operator  $\mathcal{D}_p = \mathcal{D}_p^{(p)}$  on  $C(V)$  by

$\mathcal{D}_p f(v) = d_{G;p}^{(p)}(v)f(v)$  for each function  $f \in C(V)$ , and set  $\Delta_{\mathcal{A}_{p,q}} = \mathcal{D}_p - \mathcal{A}_{p,q}$  and  $\Delta_{\mathcal{Q}_{p,q}} = I - \mathcal{Q}_{p,q}$  with the identity operator  $I$  on  $C(V)$ . We call them *combinatorial* and *probabilistic transitional*  $(p, q)$ -Laplacians, respectively. If the principal graph  $G^{(p)}$  of a Kähler graph  $G$  is regular, that is, if the degree  $d_G^{(p)}$  does not depend on the choice of vertices, then we have  $\mathcal{A}_{p,q} = d_G^{(p)}(d_G^{(p)} - 1)^{p-1} \mathcal{Q}_{p,q}$  and hence have  $\Delta_{\mathcal{A}_{p,q}} = d_G^{(p)}(d_G^{(p)} - 1)^{p-1} \Delta_{\mathcal{Q}_{p,q}}$ .

In order to see that Kähler graphs are compound graphs, we decompose these operators  $\mathcal{A}_{p,q}$ ,  $\mathcal{Q}_{p,q}$  into two operators associated with principal and auxiliary graphs. For a Kähler graph  $G = (V, E^{(p)} \cup E^{(a)})$  we define two operators  $\mathcal{A}_p^{(p)}, \mathcal{P}_p^{(p)}$  acting on  $C(V)$  by

$$\mathcal{A}_p^{(p)} f(v) = \sum_{\sigma} f(t(\sigma)), \quad \mathcal{P}_p^{(p)} f(v) = \frac{1}{d_{G;p}^{(p)}(v)} \sum_{\sigma} f(t(\sigma)),$$

where  $\sigma$  runs over the set of all  $p$ -step paths in  $(V, E^{(p)})$  without back-tracking whose origin is  $v$ . When  $p = 1$ , they are the adjacency operator  $\mathcal{A}_{G^{(p)}}$  and the transition operator  $\mathcal{P}_{G^{(p)}}$  of  $G^{(p)} = (V, E^{(p)})$ . For a  $q$ -step path  $\rho = (v_0, \dots, v_q)$  without back-tracking in the auxiliary graph  $G^{(a)} = (V, E^{(a)})$  we define its probabilistic weight  $\omega(\rho)$  by regarding it as  $(0, q)$ -bicolored path. That is, we set  $\omega(\rho) = \{d_G^{(a)}(v_0) \prod_{j=1}^{q-1} \{d_G^{(a)}(v_j) - 1\}\}^{-1}$ . We define an operator  $\mathcal{Q}_q^{(a)}$  acting on  $C(V)$  by

$$\mathcal{Q}_q^{(a)} f(v) = \sum_{\rho} \omega(\rho) f(t(\rho)),$$

where  $\rho$  runs over the set of all  $q$ -step paths in  $G^{(a)}$  without back-tracking whose origin is  $v$ . We should note  $\sum_{\rho} \omega(\rho) = 1$ . Clearly, when  $q = 1$  this operator is given as  $\mathcal{Q}_1^{(a)} f(v) = \frac{1}{d^{(a)}(v)} \sum_{w \sim_a v} f(w)$ . Hence, it coincides with the transition operator  $\mathcal{P}_{G^{(a)}}$  of  $G^{(a)}$ . Define an operator  $\mathcal{P}_q^{(a)}$  for  $G^{(a)}$  as we did for  $\mathcal{P}_p^{(p)}$ . When  $G^{(a)}$  is regular, then we have  $\mathcal{Q}_q^{(a)} = \mathcal{P}_q^{(a)}$ , but they do not coincide with each other in general.

LEMMA 1. *Let  $G = (V, E^{(p)} \cup E^{(a)})$  be a finite Kähler graph. For an arbitrary pair  $(p, q)$  of relatively prime positive integers, we have the following:*

- (1)  $d_{G;p}^{(p)}(v) = \sum_{\gamma} \omega(\gamma)$ , where  $\gamma$  runs over the set of all  $(p, q)$ -primitive bicolored paths with  $o(\gamma) = v$ ;
- (2)  $\mathcal{A}_{p,q} = \mathcal{A}_p^{(p)} \circ \mathcal{Q}_q^{(a)}$  and  $\mathcal{Q}_{p,q} = \mathcal{P}_p^{(p)} \circ \mathcal{Q}_q^{(a)}$ .

PROOF. (1) Given  $v \in V$  we denote by  $S(p, q; v)$  the set of all  $(p, q)$ -primitive bicolored paths whose origins are  $v$ . For a  $p$ -step path  $\sigma$  in the principal graph  $G^{(p)}$  which does not have back-tracking, we denote by  $S(q; \sigma)$  the set of all  $(p, q)$ -primitive bicolored paths whose first  $p$ -step subpaths coincide with  $\sigma$ . Then we find that

$$S(q; \sigma) = \left\{ \sigma \cdot \rho \left| \begin{array}{l} \rho \text{ is a } q\text{-step path in } (V, E^{(a)}) \text{ which does not} \\ \text{have back-tracking and satisfies } o(\rho) = t(\sigma) \end{array} \right. \right\}$$

Here, for paths  $\sigma = (v_0, \dots, v_p)$  and  $\rho = (w_0, \dots, w_q)$  in  $G$  satisfying  $v_p = w_0$  we set  $\sigma \cdot \rho = (v_0, \dots, v_p, w_1, \dots, w_q)$ . We have also that  $S(p, q; v)$  is the disjoint union of such sets, i.e.  $S(p, q; v) = \bigcup_{\sigma} S(q; \sigma)$ , where  $\sigma$  runs over the set of all  $p$ -step paths in  $G^{(p)}$  without back-tracking and with  $o(\sigma) = v$ . Since we defined the weights of  $q$ -step paths in  $G^{(a)}$  without back-tracking so that  $\mathcal{Q}_q^{(a)} 1 = 1$ , we get the first assertion.

(2) By definition we have

$$\begin{aligned} \mathcal{A}_{p,q} f(v) &= \sum_{\gamma \in S(p,q;v)} \omega(\gamma) f(t(\gamma)) = \sum_{\sigma} \sum_{\gamma = \sigma \cdot \rho \in S(q;\sigma)} \omega(\rho) f(t(\rho)) \\ &= \sum_{\sigma} \mathcal{Q}_q^{(a)} f(t(\sigma)) = \mathcal{A}_p^{(p)} \mathcal{Q}_q^{(a)} f(v). \end{aligned}$$

We get  $\mathcal{P}_{p,q} = \mathcal{P}_p^{(p)} \mathcal{Q}_q^{(a)}$  by the first assertion.  $\square$

As a consequence of Lemma, we see that  $\mathcal{A}_{p,q} f(v) = d_p^{(p)}(v) f$  and hence  $\Delta_{\mathcal{A}_{p,q}} f = 0$  for every constant function  $f \in C(V)$ . A graph is said to be connected if for each pair of distinct vertices there is a path joining them.

**COROLLARY 1.** *When the derived graph  $G_{p,q}$  of a Kähler graph  $G$  has  $k_{p,q}$  connected components, then its  $(p, q)$ -Laplacians  $\Delta_{\mathcal{A}_{p,q}}$  and  $\Delta_{\mathcal{P}_{p,q}}$  have null eigenvalues. Their multiplicities are the number  $k_{p,q}$  of connected components of  $G_{p,q}$ .*

**4. Eigenvalues of Finite Regular Kähler Graphs** In this section we study eigenvalues of  $(p, q)$ -Laplacians for finite regular Kähler graphs. We call a Kähler graph  $G = (V, E^{(p)} \cup E^{(a)})$  *regular* if both of its principal and its auxiliary graphs are regular, that is, the degrees  $d_G^{(p)}$  and  $d_G^{(a)}$  do not depend on the choice of vertices.

Given a positive integer  $d (\geq 2)$ , we define a sequence  $\{F_n(t; d)\}_{n=0}^{\infty}$  of monic polynomials by the relations

$$\begin{cases} F_{n+1}(t; d) = tF_n(t; d) - (d-1)F_{n-1}(t; d) & (n \geq 2), \\ F_0(t; d) = 1, F_1(t; d) = t, F_2(t; d) = t^2 - d. \end{cases}$$

Therefore we have  $F_3(t; d) = t^3 - (2d-1)t$  and  $F_4(t; d) = t^4 - (3d-2)t^2 + d(d-1)$ , for example. The degree of the polynomial  $F_n(t; d)$  is  $n$ , and it contains only terms of odd degree if  $n$  is odd and contains only terms of even degree if  $n$  is even. These polynomials satisfy  $F_n(d; d) = d(d-1)^{n-1}$  for  $n \geq 1$  and  $F_{2k-1}(0; d) = 0$ ,  $F_{2k}(0; d) = (-1)^k d(d-1)^{k-1}$  for  $k \geq 1$ .

For a finite graph  $H = (W, F)$ , we denote by  $n_H$  the cardinality of the set  $W$  of vertices. We here recall the adjacency and transition operators by  $n$ -step paths on a regular graphs.

**LEMMA 2.** *Let  $H = (W, F)$  be a regular graph of degree  $d_H$ .*

- (1) The adjacency operator  $\mathcal{A}_n$  by  $n$ -step paths on  $H$  without back-tracking is given as  $F_n(\mathcal{A}_H; d_H)$ .
- (2) The transition operator  $\mathcal{P}_n$  by  $n$ -step paths on  $H$  without back-tracking is given as  $d_H^{-1}(d_H - 1)^{1-n}F_n(\mathcal{A}_H; d_H)$ .

Here, for a positive  $k$  the operator  $\mathcal{A}_H^k$  means the  $k$ th-composition  $\mathcal{A}_H \circ \cdots \circ \mathcal{A}_H$  and  $\mathcal{A}_H^0 = I$ .

PROOF. As  $\mathcal{A}_1 = \mathcal{A}_H$ , it is trivial that  $\mathcal{A}_1 = F_1(\mathcal{A}_H; d_H)$ . A 2-step path on  $H$  with back-tracking is of the form  $(w_0, w_1, w_0)$  with  $w_0 \sim w_1$ . Since  $H$  is regular of degree  $d_H$ , we see the adjacency by 2-step paths without back-tracking is expressed by the operator  $\mathcal{A}_2 = \mathcal{A}_H^2 - d_H I$ .

For  $n \geq 3$ , we take an  $n$ -step path  $(w_0, \dots, w_n)$  without back-tracking. When we consider a sequence of vertices  $(w_0, \dots, w_n, w)$ , we see that it is an  $(n+1)$ -step path if and only if  $w \sim w_n$  and that under this condition it is without back-tracking if and only if  $w \neq w_{n-1}$ . Since  $w_{n-2} \neq w_n$ , we find that the adjacency operators satisfy  $\mathcal{A}_{n+1} = \mathcal{A}_n \mathcal{A}_H - (d_H - 1)\mathcal{A}_{n-1}$ . Since  $F_n(\mathcal{A}_H; d_H)$  and  $\mathcal{A}_H$  are commutative, we find  $\mathcal{A}_n = F_n(\mathcal{A}_H; d_H)$ , which is the first assertion. Since we can not go back, the cardinality of  $n$ -step paths emanating from a vertex is  $d_H(d_H - 1)^{n-1}$ . Hence we get the second assertion.  $\square$

As a consequence of Lemmas 1, 2 and of a property of eigenvalues of ordinary regular graphs, we have the following.

THEOREM 1. Let  $G = (V, E^{(p)} \cup E^{(a)})$  be a finite regular Kähler graph. Suppose its adjacency operators  $\mathcal{A}_{G^{(p)}}$ ,  $\mathcal{A}_{G^{(a)}}$  of the principal and the auxiliary graphs are commutative ( $\mathcal{A}_{G^{(p)}} \circ \mathcal{A}_{G^{(a)}} = \mathcal{A}_{G^{(a)}} \circ \mathcal{A}_{G^{(p)}}$ ). We denote the eigenvalues of  $\mathcal{A}_{G^{(p)}}$  by  $\mu_j$  ( $j = 1, \dots, n_G$ ) and those of  $\mathcal{A}_{G^{(a)}}$  by  $\nu_j$  ( $j = 1, \dots, n_G$ ), where  $\mu_j$  and  $\nu_j$  have common eigenfunctions. Then the eigenvalues of the combinatorial  $(p, q)$ -Laplacian  $\Delta_{\mathcal{A}_{p,q}}$  are

$$d_G^{(p)}(d_G^{(p)} - 1)^{p-1} - \frac{F_p(\mu_j; d_G^{(p)})F_q(\nu_j; d_G^{(a)})}{d_G^{(a)}(d_G^{(a)} - 1)^{q-1}}, \quad j = 1, \dots, n_G.$$

PROOF. Since these adjacency operators  $\mathcal{A}_{G^{(p)}}$ ,  $\mathcal{A}_{G^{(a)}}$  are symmetric and simultaneously diagonalizable, there exist functions  $f_j \in C(V)$ ,  $j = 1, 2, \dots, n_G$  satisfying  $f_j \neq 0$ ,  $\mathcal{A}^{(p)}f_j = \mu_j f_j$  and  $\mathcal{A}^{(a)}f_j = \nu_j f_j$  with their eigenvalues  $\mu_j$  and  $\nu_j$ . As  $G$  is regular, we have

$$\mathcal{A}_p^{(p)} = F_p(\mathcal{A}^{(p)}; d_G^{(p)}) \quad \text{and} \quad \mathcal{Q}_q^{(a)} = \frac{1}{d_G^{(a)}(d_G^{(a)} - 1)^{q-1}} F_q(\mathcal{A}^{(a)}; d_G^{(a)}).$$

Thus, by Lemma 1, we have

$$\begin{aligned} \mathcal{A}_{p,q} f_j &= \frac{1}{d_G^{(a)}(d_G^{(a)} - 1)^{q-1}} F_p(\mathcal{A}_p^{(p)}; d_G^{(p)}) F_q(\mathcal{A}_q^{(a)}; d_G^{(a)}) f_j \\ &= \frac{F_p(\mu_j; d_G^{(p)}) F_q(\nu_j; d_G^{(a)})}{d_G^{(a)}(d_G^{(a)} - 1)^{q-1}} f_j. \end{aligned}$$

We hence get the conclusion.  $\square$

**COROLLARY 2.** *Let  $G$  be a finite regular Kähler graph. Under the same assumption as in Theorem 1, its eigenvalues of the probabilistic transition  $(p, q)$ -Laplacian  $\Delta_{Q_{p,q}}$  are*

$$1 - \frac{F_p(\mu_j; d_G^{(p)})F_q(\nu_j; d_G^{(a)})}{d_G^{(p)}d_G^{(a)}(d_G^{(p)} - 1)^{p-1}(d_G^{(a)} - 1)^{q-1}}, \quad j = 1, \dots, n_G.$$

*Remark.* If the adjacency operators  $\mathcal{A}_{G^{(p)}}$ ,  $\mathcal{A}_{G^{(a)}}$  of a regular Kähler graph  $G$  are commutative, then its  $(p, q)$ -adjacency operator  $\mathcal{A}_{p,q}$  and its  $(p, q)$ -probabilistic transition operator  $\mathcal{Q}_{p,q}$  are symmetric.

We now apply Theorem 1 to some typical examples of Kähler graphs. For a finite graph  $G = (V, E)$  whose degree satisfies  $2 \leq \min_{v \in V} d_G(v) \leq \max_{v \in V} d_G(v) \leq n_G - 3$ , we take its complement graph  $G^c = (V, E^c)$ . Here, we define that two distinct vertices are adjacent in  $G^c$  if and only if they are not adjacent in  $G$ . Then we obtain a Kähler graph  $G^K = (V, E \cup E^c)$ . We say this to be the *complement-filled* Kähler graph of  $G$ . Clearly when  $G$  is regular then  $G^K$  is a regular Kähler graph.

**PROPOSITION 1.** *Let  $G = (V, E)$  be a connected regular finite graph whose degree satisfies  $2 \leq d_G \leq n_G - 3$ . If we denote by  $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_{n_G}$  the eigenvalues of the combinatorial Laplacian  $\Delta_{\mathcal{A}_G}$  of  $G$ , we see the eigenvalues of the combinatorial  $(p, q)$ -Laplacian  $\Delta_{\mathcal{A}_{p,q}}$  of  $G^K$  are 0 and*

$$d_G(d_G - 1)^{p-1} - \frac{F_p(d_G - \lambda_j; d_G)F_q(\lambda_j - d_G - 1; n_G - d_G - 1)}{(n_G - d_G - 1)(n_G - d_G - 2)^{q-1}}$$

for  $j = 2, \dots, n_G$ .

**PROOF.** We put  $H = G^K$ . Since  $\mathcal{A}_{G^c} = \mathcal{M} - I - \mathcal{A}_G$  with the operator  $\mathcal{M}$  defined by  $\mathcal{M}f(v) = \sum_{w \in V} f(w)$ , we see that  $\mathcal{A}_{H^{(p)}} = \mathcal{A}_G$  and  $\mathcal{A}_{H^{(a)}} = \mathcal{A}_{G^c}$  are commutative. If a nonzero function  $f_j$  satisfies  $\Delta_{\mathcal{A}_G} f_j = \lambda_j f_j$ , we have  $\mathcal{A}_G f_j = (d_G - \lambda_j) f_j$ . Moreover, as  $G$  is connected, we see  $f_1$  is constant and  $f_j$  ( $j \geq 2$ ) is orthogonal to  $f_1$ , that is,  $\sum_{v \in V} f_1(v) f_j(v) = 0$ . Thus we find that

$$\mathcal{A}_{G^c} f_j = \begin{cases} (n_G - d_G - 1) f_j, & j = 1, \\ -f_j - \mathcal{A}_G f_j = (\lambda_j - d_G - 1) f_j, & j \geq 2 \end{cases}$$

We hence get the conclusion by Theorem 1.  $\square$

Given two finite graphs  $G = (V, E)$  and  $H = (W, F)$  we define Kähler graphs of Cartesian product type  $G \widehat{\square} H$ , of strong product type  $G \widehat{\boxtimes} H$ , of semi-tensor product type  $G \widehat{\otimes} H$ , and of lexicographic product type  $G \triangleright H$  as follows:

- (i) the set of vertices is  $V \times W$ ;

- (ii) two vertices  $(v, w), (v', w') \in V \times W$  are adjacent in the principal graph if and only if  $v, v'$  are adjacent in  $G$  and  $w = w'$ ;
- (iii) two vertices  $(v, w), (v', w') \in V \times W$  are adjacent in the auxiliary graph if and only if they satisfy the following conditions;
  - (a) for  $G\widehat{\cap}H$ ,  $v = v'$  and  $w, w'$  are adjacent in  $H$ ,
  - (b) for  $G\widehat{\boxtimes}H$ , either  $v = v'$  and  $w, w'$  are adjacent in  $H$ , or  $v, v'$  are adjacent in  $G$  and  $w, w'$  are adjacent in  $H$ ,
  - (c) for  $G\widehat{\otimes}H$ ,  $v, v'$  are adjacent in  $G$  and  $w, w'$  are adjacent in  $H$ ,
  - (d) for  $G\triangleright H$ ,  $w, w'$  are adjacent in  $H$ .

Then it is clear that their adjacency operators  $\mathcal{A}^{(p)}, \mathcal{A}^{(a)}$  of principal and auxiliary graphs are commutative. When both  $G$  and  $H$  are finite graphs, the principal and the auxiliary degrees of these graphs of product types at a vertex  $(v, w) \in V \times W$  are given as

$$\begin{aligned}
 d_{(G\widehat{\cap}H)^{(p)}}(v, w) &= d_{(G\widehat{\boxtimes}H)^{(p)}}(v, w) = d_{(G\widehat{\otimes}H)^{(p)}}(v, w) = d_{(G\triangleright H)^{(p)}}(v, w) = d_G(v), \\
 d_{(G\widehat{\cap}H)^{(a)}}(v, w) &= d_H(w), \quad d_{(G\widehat{\boxtimes}H)^{(a)}}(v, w) = d_H(w)(d_G(v) + 1), \\
 d_{(G\widehat{\otimes}H)^{(a)}}(v, w) &= d_G(v)d_H(w), \quad d_{(G\triangleright H)^{(a)}}(v, w) = d_H(w)n_G.
 \end{aligned}$$

Thus, if both  $G$  and  $H$  are regular, then these Kähler graphs of product types are regular Kähler graphs. Following [10], we can calculate eigenvalues of  $(p, q)$ -Laplacians of these regular Kähler graphs.

**PROPOSITION 2.** *Let  $G = (V, E)$  and  $H = (W, F)$  are regular finite graphs. We denote by  $0 = \lambda_1 \leq \dots \leq \lambda_{n_G}$  the eigenvalues of the combinatorial Laplacian  $\Delta_{\mathcal{A}_G}$  of  $G$  and by  $0 = \rho_1 \leq \dots \leq \rho_{n_H}$  the eigenvalues of the combinatorial Laplacian  $\Delta_{\mathcal{A}_H}$  of  $H$ .*

- (1) *The eigenvalues of the combinatorial  $(p, q)$ -Laplacians of  $G\widehat{\cap}H$  are*

$$d_G(d_G - 1)^{p-1} - \frac{F_p(d_G - \lambda_j; d_G)F_q(d_H - \rho_\alpha; d_H)}{d_H(d_H - 1)^{q-1}}.$$

- (2) *The eigenvalues of the combinatorial  $(p, q)$ -Laplacians of  $G\widehat{\boxtimes}H$  are*

$$d_G(d_G - 1)^{p-1} - \frac{F_p(d_G - \lambda_j; d_G)F_q((d_G - \lambda_j + 1)(d_H - \rho_\alpha); d_H(d_G + 1))}{d_H(d_G + 1)(d_G d_H + d_H - 1)^{q-1}}.$$

- (3) *The eigenvalues of the combinatorial  $(p, q)$ -Laplacians of  $G\widehat{\otimes}H$  are*

$$d_G(d_G - 1)^{p-1} - \frac{F_p(d_G - \lambda_j; d_G)F_q((d_G - \lambda_j)(d_H - \rho_\alpha); d_G d_H)}{d_G d_H (d_G d_H - 1)^{q-1}}.$$

PROOF. By taking characteristic functions of  $V$  and  $W$  as bases, we represent adjacency operators  $\mathcal{A}_G, \mathcal{A}_H$  by adjacency matrices  $A_G = (a_{ij}^G)$ ,  $A_H = (a_{\alpha\beta}^H)$ , respectively. Then the common adjacency matrix  $A^{(p)} = (a_{(i,\alpha),(j,\beta)}^{(p)})$  of principal graphs of the three Kähler graphs of product types are given as  $a_{(i,\alpha),(j,\beta)}^{(p)} = a_{ij}^G \delta_{\alpha\beta}$  where  $\delta_{\alpha\beta}$  denotes the Kronecker delta. We take eigenfunctions  $f_i$  and  $g_\alpha$  which are associated with  $\lambda_i$  and  $\rho_\alpha$ , respectively, and define a function  $\varphi_{i\alpha}$  on  $V \times W$  by  $\varphi_{i\alpha}(v, w) = f_i(v)g_\alpha(w)$ . By the expression of  $A^{(p)}$  we find that the common adjacency operator  $\mathcal{A}^{(p)}$  of the principal graphs satisfies  $\mathcal{A}^{(p)}\varphi_{i\alpha} = (d_G - \lambda_i)\varphi_{i\alpha}$ . Hence its eigenvalues are  $d_G - \lambda_i$  ( $i = 1, \dots, n_G$ ). We note that each of them are counted  $n_H$  times.

The adjacency matrices of the auxiliary graphs of  $G\widehat{\square}H$ ,  $G\widehat{\boxtimes}H$ ,  $G\widehat{\otimes}H$  are given as  $(\delta_{ij}a_{\alpha\beta}^H)$ ,  $((a_{ij}^G + \delta_{ij})a_{\alpha\beta}^H)$ ,  $(a_{ij}^G a_{\alpha\beta}^H)$ , respectively. Hence for the three Kähler graphs of product type we find that adjacency operators of their principal graphs and their auxiliary graphs are commutative. By the expression of the adjacency operator  $\mathcal{A}_{(G\widehat{\square}H)^{(a)}}$  of the auxiliary graph of  $G\widehat{\square}H$  we find that it satisfies  $\mathcal{A}_{(G\widehat{\square}H)^{(a)}}\varphi_{i\alpha} = (d_H - \rho_\alpha)\varphi_{i\alpha}$  and that its eigenvalues are given as  $d_H - \rho_\alpha$  ( $\alpha = 1, \dots, n_H$ ) each of which are counted  $n_G$  times. Similarly, the adjacency operator  $\mathcal{A}_{(G\widehat{\boxtimes}H)^{(a)}}$  of the auxiliary graph  $(G\widehat{\boxtimes}H)^{(a)}$  satisfies  $\mathcal{A}_{(G\widehat{\boxtimes}H)^{(a)}}\varphi_{i\alpha} = (d_G - \lambda_j + 1)(d_H - \rho_\alpha)\varphi_{i\alpha}$ , and the adjacency operator  $\mathcal{A}_{(G\widehat{\otimes}H)^{(a)}}$  of the auxiliary graph  $(G\widehat{\otimes}H)^{(a)}$  satisfies  $\mathcal{A}_{(G\widehat{\otimes}H)^{(a)}}\varphi_{i\alpha} = (d_G - \lambda_j)(d_H - \rho_\alpha)\varphi_{i\alpha}$ . We get our result by applying Theorem 1 with the common eigenfunctions  $\varphi_{i\alpha}$ .  $\square$

For a Kähler graph of lexicographic product type we can calculate its eigenvalues for  $(p, q)$ -Laplacians a bit more directly.

PROPOSITION 3. *Let  $G = (V, E)$  and  $H = (W, F)$  are finite graphs. Suppose  $H$  is regular. We denote by  $0 = \rho_1 \leq \dots \leq \rho_{n_H}$  the eigenvalues of the combinatorial Laplacian  $\Delta_{\mathcal{A}_H}$  of  $H$ .*

- (1) *When  $q$  is odd, the eigenvalues of the probabilistic transitional  $(p, q)$ -Laplacian of  $G \triangleright H$  are 1 and*

$$1 - F_q(n_G(d_H - \rho_\alpha); n_G d_H) / \{n_G d_H (n_G d_H - 1)^{q-1}\} \quad (\alpha = 1, \dots, n_H),$$

*where the former 1 appears  $(n_G - 1)n_H$  times.*

- (2) *We denote by  $0 = \lambda_1 = \dots = \lambda_{k_G} < \lambda_{k_G+1} \leq \dots \leq \lambda_{n_G}$  the eigenvalues of the combinatorial Laplacian  $\Delta_{\mathcal{A}_G}$  of  $G$ , where  $k_G$  is the number of connected components of  $G$ . When  $G$  is regular and  $q$  is even, the eigenvalues of the probabilistic transitional  $(p, q)$ -Laplacian of  $G \triangleright H$  are*

$$1 - \frac{F_q(n_G(d_H - \rho_\alpha); n_G d_H)}{n_G d_H (n_G d_H - 1)^{q-1}} \quad (\alpha = 1, \dots, n_H), \quad 1 - \frac{(-1)^{q/2}}{(n_G d_H - 1)^{q/2}},$$

$$1 - \frac{(-1)^{q/2} F_p(d_G - \lambda_i; d_G)}{d_G (d_G - 1)^{p-1} (n_G d_H - 1)^{q/2}} \quad (i = k_G + 1, \dots, n_G).$$

Here, the eigenvalue of the second type appears  $n_H(k_G - 1)$  times, and each eigenvalue of the third type appears  $n_H$  times.

PROOF. We denote by  $A_{G;p} = (a_{ij;p}^G)$  the  $p$ -step adjacency matrix of  $G$ . That is, we define  $a_{ij;p}^G$  as the cardinality of the set of  $p$ -step paths on  $G$  without back-tracking which join vertices  $v_i$  and  $v_j$ . We set  $d_{G;p}(v)$  the cardinality of the set of all  $p$ -step paths without back-tracking emanating from  $v$ . We hence have  $d_{G;p}(v_i) = \sum_j a_{ij;p}^G$ . Since the principal graph of  $G \triangleright H$  is isomorphic to a disjoint union of  $n_H$ -copies of  $G$ , if we express the operator  $\mathcal{A}_p^{(p)}$  of  $G \triangleright H$  by a matrix  $(a_{(i,\alpha),(j,\beta);p}^{(p)})$ , we have  $a_{(i,\alpha),(j,\beta);p}^{(p)} = a_{ij;p}^G \delta_{\alpha\beta}$ .

Next we study the auxiliary graph of  $G \triangleright H$ . We denote by  $A_H$  the matrix representation of the adjacency operator  $\mathcal{A}_H$  of  $H$ . The adjacency operator of the auxiliary graph  $(G \triangleright H)^{(a)}$  is expressed by the matrix

$$A^{(a)} = \begin{pmatrix} A_H & \cdots & A_H \\ \vdots & & \vdots \\ A_H & \cdots & A_H \end{pmatrix},$$

where components of  $A^{(a)}$  are expressed according to lexicographical order. Clearly its  $k$ th power is given as

$$(A^{(a)})^k = \begin{pmatrix} n_G^{k-1} A_H^k & \cdots & n_G^{k-1} A_H^k \\ \vdots & & \vdots \\ n_G^{k-1} A_H^k & \cdots & n_G^{k-1} A_H^k \end{pmatrix}.$$

As  $H$  is regular, we see this graph  $(G \triangleright H)^{(a)}$  is regular of degree  $n_G d_H$ . We hence find by Lemma 2 that the  $q$ -step adjacency matrix of  $(G \triangleright H)^{(a)}$  is given by  $F_q(A^{(a)}; n_G d_H)$ . We show below the following by mathematical induction:

$$F_q(A^{(a)}; n_G d_H) = \begin{cases} \begin{pmatrix} N_q & \cdots & N_q \\ \vdots & & \vdots \\ N_q & \cdots & N_q \end{pmatrix}, & \text{when } q \text{ is odd,} \\ \begin{pmatrix} N_q & \cdots & N_q \\ \vdots & & \vdots \\ N_q & \cdots & N_q \end{pmatrix} & \text{when } q \text{ is even.} \\ + (-1)^{q/2} n_G d_H (n_G d_H - 1)^{(q/2)-1} I_{n_G n_H}, & \end{cases}$$

where

$$N_q = \begin{cases} n_G^{-1} F_q(n_G A_H; n_G d_H), & \text{when } q \text{ is odd,} \\ n_G^{-1} F_q(n_G A_H; n_G d_H) & \text{when } q \text{ is even,} \\ - (-1)^{q/2} d_H (n_G d_H - 1)^{(q/2)-1} I_{n_H}, & \end{cases}$$

and  $I_k$  denotes the identity matrix of size  $k$ . Since  $F_1(A^{(a)}; n_G d_H) = A^{(a)}$  and  $F_2(A^{(a)}; n_G d_H) = (A^{(a)})^2 - n_G d_H I_{n_G n_H}$ , these satisfy the above. We suppose the above expression holds for  $1 \leq q \leq 2\ell$  ( $\ell \geq 1$ ). As we have

$$F_{q+1}(A^{(a)}; n_G d_H) = F_q(A^{(a)}; n_G d_H)A^{(a)} - (n_G d_H - 1)F_{q-1}(A^{(a)}; n_G d_H),$$

we find that

$$\begin{aligned} & F_{2\ell+1}(A^{(a)}; n_G d_H) \\ &= \begin{pmatrix} n_G N_{2\ell} A_H & \cdots & n_G N_{2\ell} A_H \\ \vdots & & \vdots \\ n_G N_{2\ell} A_H & \cdots & n_G N_{2\ell} A_H \end{pmatrix} + (-1)^\ell n_G d_H (n_G d_H - 1)^{\ell-1} A^{(a)} \\ &\quad - (n_G d_H - 1) \begin{pmatrix} N_{2\ell-1} & \cdots & N_{2\ell-1} \\ \vdots & & \vdots \\ N_{2\ell-1} & \cdots & N_{2\ell-1} \end{pmatrix}. \end{aligned}$$

As we have

$$\begin{aligned} & n_G N_{2\ell} A_H + (-1)^\ell n_G d_H (n_G d_H - 1)^{\ell-1} A_H - (n_G d_H - 1) N_{2\ell-1} \\ &= n_G^{-1} \{ F_{2\ell}(n_G A_H; n_G d_H) n_G A_H - (n_G d_H - 1) F_{2\ell-1}(n_G A_H; n_G d_H) \} \\ &= n_G^{-1} F_{2\ell+1}(n_G A_H; n_G d_H), \end{aligned}$$

we find that the expression of  $F_q(A^{(a)}; n_G d_H)$  holds for  $q = 2\ell + 1$ . We therefore have

$$\begin{aligned} & F_{2\ell+2}(A^{(a)}; n_G d_H) \\ &= \begin{pmatrix} n_G N_{2\ell+1} A_H & \cdots & n_G N_{2\ell+1} A_H \\ \vdots & & \vdots \\ n_G N_{2\ell+1} A_H & \cdots & n_G N_{2\ell+1} A_H \end{pmatrix} \\ &\quad - (n_G d_H - 1) \begin{pmatrix} N_{2\ell} & \cdots & N_{2\ell} \\ \vdots & & \vdots \\ N_{2\ell} & \cdots & N_{2\ell} \end{pmatrix} - (-1)^\ell n_G d_H (n_G d_H - 1)^\ell I_{n_G n_H}. \end{aligned}$$

As we have

$$\begin{aligned} & n_G N_{2\ell+1} A_H - (n_G d_H - 1) N_{2\ell} \\ &= n_G^{-1} \{ F_{2\ell+1}(n_G A_H; n_G d_H) n_G A_H - (n_G d_H - 1) F_{2\ell}(n_G A_H; n_G d_H) \} \\ &\quad - (-1)^{\ell+1} d_H (n_G d_H - 1)^\ell I_{n_H} \\ &= n_G^{-1} F_{2\ell+2}(n_G A_H; n_G d_H) - (-1)^{\ell+1} d_H (n_G d_H - 1)^\ell I_{n_H}, \end{aligned}$$

we find that the expression of  $F_q(A^{(a)}; n_G d_H)$  also holds for  $q = 2\ell + 2$ . Thus we get the form of  $F_q(A^{(a)}; n_G d_H)$  by induction.

If we set  $N_q = (b_{\alpha\beta;q})$ , the operator  $\mathcal{Q}_q^{(a)}$  of  $G \triangleright H$  is represented by a matrix as

$$\begin{cases} (c_{\alpha\beta;q}), & \text{when } q \text{ is odd,} \\ (c_{\alpha\beta;q}) + (-1)^{q/2}(n_G d_H - 1)^{-q/2}(\delta_{ij}\delta_{\alpha\beta}), & \text{when } q \text{ is even,} \end{cases}$$

where  $c_{\alpha\beta;q} = b_{\alpha\beta;q}/\{n_G d_H(n_G d_H - 1)^{q-1}\}$ .

When  $q$  is odd, the  $(p, q)$ -probabilistic transition operator  $\mathcal{Q}_{p,q}$  is represented by a matrix  $Q_{p,q}$  as

$$Q_{p,q} = (d_{G;p}(i)^{-1} a_{ij;p}^G \delta_{\alpha\beta})(c_{\alpha\beta;q}) = \left( d_{G;p}(i)^{-1} \left( \sum_j a_{ij;p}^G \right) c_{\alpha\beta;q} \right) = (c_{\alpha\beta;q}),$$

where  $d_{G;p}(i)$  denotes the cardinality of  $p$ -step paths on  $G$  without back-tracking emanating from  $v_i$ . We define functions  $\epsilon_i$  on  $V$  by  $\epsilon_1(v) = 1$  for all  $v \in V$ , and by  $\epsilon_i = \delta_{v_1} - \delta_{v_i}$  for  $2 \leq i \leq n_G$ . For a nonzero function  $g_\alpha$  on  $W$  satisfying  $\Delta_{\mathcal{A}_H} g = \rho_\alpha g_\alpha$  we define a function  $\psi_{i,\alpha}$  on  $V \times W$  by  $\psi_{i,\alpha}(v, w) = \epsilon_i(v)g_\alpha(w)$ . By the matrix representation of  $\mathcal{Q}_{p,q}$  we have

$$\mathcal{Q}_{p,q}\psi_{1,\alpha} = \frac{F_q(n_G(d_H - \rho_\alpha); n_G d_H)}{n_G d_H(n_G d_H - 1)^{q-1}}\psi_{1,\alpha}, \quad \mathcal{Q}_{p,q}\psi_{i,\alpha} = 0 \quad (i = 2, \dots, n_G).$$

Thus we get the first assertion.

When  $G$  is regular and  $q$  is even, we have

$$\begin{aligned} Q_{p,q} &= \frac{1}{d_G(d_G - 1)^{p-1}} (a_{ij;p}^G \delta_{\alpha\beta}) \left( c_{\alpha\beta;q} + (-1)^{q/2}(n_G d_H - 1)^{-q/2} \delta_{ij}\delta_{\alpha\beta} \right) \\ &= \left( c_{\alpha\beta;q} + \frac{(-1)^{q/2}}{d_G(d_G - 1)^{p-1}(n_G d_H - 1)^{q/2}} a_{ij;p}^G \delta_{\alpha\beta} \right). \end{aligned}$$

We decompose  $G$  into connected components  $G_j = (V_j, E_j)$  ( $j = 1, \dots, k_G$ ) and denote by  $n_j$  the cardinality of  $V_j$ . We set  $f_1 \equiv 1$  and  $f_i = n_1\delta_{V_i} - n_i\delta_{V_1}$  for  $i = 2, \dots, k_G$ , where  $\delta_{V_i}$  denotes the characteristic function of  $V_i$  in  $V$ . For  $i = k_G + 1, \dots, n_G$  we take a nonzero function  $f_i$  on  $V$  satisfying  $\Delta_{\mathcal{A}_G} f_i = \lambda_i f_i$ . Then we find that  $\sum_{v \in V} f_i(v) = 0$  for  $i = 2, \dots, k_G$  by definition and that  $\sum_{v \in V} f_i(v) = 0$  for  $i = k_G + 1, \dots, n_G$  because they are orthogonal to  $f_1, \dots, f_{k_G}$ . Taking nonzero functions  $g_\alpha$  on  $W$  satisfying  $\Delta_{\mathcal{A}_H} g_\alpha = \rho_\alpha g_\alpha$ , we define a function  $\varphi_{i\alpha}$  on  $V \times W$  by  $\varphi_{i\alpha}(v, w) = f_i(v)g_\alpha(w)$ . By the matrix representation of  $\mathcal{Q}_{p,q}$  we have

$$\mathcal{Q}_{p,q}\varphi_{i,\alpha} = \begin{cases} \frac{F_q(n_G(d_H - \rho_\alpha); n_G d_H)}{n_G d_H(n_G d_H - 1)^{q-1}} \varphi_{i,\alpha}, & i = 1, \\ \frac{(-1)^{q/2} F_p(d_G - \lambda_i; d_G)}{d_G(d_G - 1)^{p-1}(n_G d_H - 1)^{q/2}} \varphi_{i,\alpha}, & i = 2, \dots, n_G, \end{cases}$$

because  $d_{G;p} = d_G(d_G - 1)^{p-1}$  and the  $p$ -step adjacency operator  $\mathcal{A}_{G;p}$  of  $G$  which is represented by the matrix  $A_{G;p}$  satisfies  $\mathcal{A}_{G;p}f_i = F_p(d_G - \lambda_i; d_G)f_i$ . As  $F_p(d_G; d_G) = d_G(d_G - 1)^{p-1}$ , we get the second assertion.  $\square$

*Remark.* When  $G$  is not regular, the adjacency operators of the principal and the auxiliary graphs of  $G \triangleright H$  are not commutative in general. When  $G$  is regular, they are commutative. Hence we can apply Theorem 1 with common eigenfunctions  $\varphi_{i\alpha}$  in the above proof.

For a graph  $G = (V, E)$  we denote by  $E_p$  the set of all  $p$ -step paths without back-tracking, and define a new graph  $G_p = (V, E_p)$ . By the proof of Proposition 3 we obtain the following.

PROPOSITION 4. *Let  $G$  be a finite graph and  $H$  be a finite regular graph. We denote by  $0 = \rho_1 \leq \dots \leq \rho_{n_H}$  the eigenvalues of the combinatorial Laplacian  $\Delta_{\mathcal{A}_H}$  of  $H$ , and denote by  $0 = \eta_{1;p} = \dots = \eta_{k_{G_p};p} < \eta_{k_{G_p}+1;p} \leq \dots \leq \eta_{n_G;p}$  the eigenvalues of the transitional Laplacian of  $G_p$ . When  $q$  is even, the eigenvalues of the probabilistic transitional  $(p, q)$ -Laplacian of  $G \triangleright H$  are*

$$1 - \frac{F_q(n_G(d_H - \rho_\alpha); n_G d_H)}{n_G d_H (n_G d_H - 1)^{q-1}} \quad (\alpha = 1, \dots, n_H), \quad 1 - \frac{(-1)^{q/2}}{(n_G d_H - 1)^{q/2}},$$

$$1 - \frac{(-1)^{q/2}(1 - \eta_{i;p})}{(n_G d_H - 1)^{q/2}} \quad (i = k_{G_p} + 1, \dots, n_G).$$

Here, the eigenvalue of the second type appears  $n_H(k_{G_p} - 1)$  times, and each eigenvalue of the third type appears  $n_H$  times.

Two ordinary finite graphs are said to be combinatorially isospectral (resp. transitionary isospectral) if their combinatorial Laplacians (resp. transitional Laplacians) have the same eigenvalues by taking account with their multiplicities. It is well known that there exist many pairs of regular graphs which are isospectral (see [4, 6], for example). We note that for two regular graphs they are combinatorially isospectral if and only if they are transitionary isospectral. Hence we just say that they are isospectral.

We here study the isospectral property for Kähler graphs. For a pair of finite Kähler graphs we shall call them  $(p, q)$ -combinatorially isospectral (resp.  $(p, q)$ -probabilistic transitionary isospectral) if they satisfy the following conditions:

- (i) their combinatorial  $(p, q)$ -Laplacians (resp. probabilistic transitional  $(p, q)$ -Laplacians) have the same eigenvalues by taking account of their multiplicities,
- (ii) their principal graphs are combinatorially (resp. transitionary) isospectral.

It is clear that these notions that two Kähler graphs are  $(p, q)$ -combinatorially and  $(p, q)$ -probabilistic transitionary isospectral are equivalent to each other if their principal graphs are regular. In this case, we just say that they are  $(p, q)$ -isospectral. As a consequence of Theorem 1 we have the following.

**THEOREM 2.** *Let  $G_1 = (V_1, E_1^{(p)} \cup E_1^{(a)})$  and  $G_2 = (V_2, E_2^{(p)} \cup E_2^{(a)})$  be two regular Kähler graphs having the same principal and auxiliary degrees (i.e.  $d_{G_1}^{(p)} = d_{G_2}^{(p)}$  and  $d_{G_1}^{(a)} = d_{G_2}^{(a)}$ ). Suppose their adjacency operators  $\mathcal{A}_{G_i}^{(p)}, \mathcal{A}_{G_i}^{(a)}$  of principal and auxiliary graphs are commutative ( $\mathcal{A}_{G_i}^{(p)} \circ \mathcal{A}_{G_i}^{(a)} = \mathcal{A}_{G_i}^{(a)} \circ \mathcal{A}_{G_i}^{(p)}, i = 1, 2$ ). If their principal graphs  $(V_1, E_1^{(p)}), (V_2, E_2^{(p)})$  are isospectral and if their auxiliary graphs  $(V_1, E_1^{(a)}), (V_2, E_2^{(a)})$  are also isospectral, then they are  $(p, q)$ -isospectral for an arbitrary pair  $(p, q)$  of relatively prime positive integers.*

The commutativity condition on adjacency operators is a strong condition. Still, we have many examples of isospectral pairs of regular Kähler graphs.

Since there are many isospectral pairs of finite regular graphs of the same degree, the operation of complement-filling is a way of constructing isospectral pairs of Kähler graphs.

**COROLLARY 3.** *If two finite regular graphs  $G_1, G_2$  are isospectral and have the same degrees, then their complement-filled Kähler graphs are  $(p, q)$ -isospectral for an arbitrary  $(p, q)$  of relatively prime positive integers.*

**EXAMPLE 1.** We take two regular graphs  $G_1, G_2$  of degree 4 as in Figures 1 and 2. It is known that they are isospectral. Hence their complement-filled Kähler graphs  $G_1^K, G_2^K$  are  $(p, q)$ -isospectral. We list here the eigenvalues of their combinatorial and transitional Laplacians and those of their  $(1, 1)$ -,  $(2, 1)$ -,  $(1, 2)$ -combinatorial and probabilistic transitional Laplacians:

$\Delta_{\mathcal{A}}$	$0, 4 - \sqrt{5}, (9 - \sqrt{17})/2, 3, 5, 5, 5, 5, 4 + \sqrt{5}, (9 + \sqrt{17})/2$
$\Delta_{\mathcal{A}_{1,1}}$	$0, 4, 4, 4, 4, 22/5, (25 - \sqrt{5})/5, 24/5, 24/5, (25 + \sqrt{5})/5$
$\Delta_{\mathcal{A}_{2,1}}$	$0, 54/5, 56/5, 56/5, (61 - \sqrt{5})/5, 12, 12, 12, 12, (61 + \sqrt{5})/5$
$\Delta_{\mathcal{A}_{1,2}}$	$0, (70 - \sqrt{5})/20, (70 + \sqrt{5})/20, (151 - \sqrt{17})/40,$ $15/4, 15/4, 15/4, 15/4, (151 + \sqrt{17})/40, 81/20$

$\Delta_{\mathcal{P}}$	$0, (4 - \sqrt{5})/4, (9 - \sqrt{17})/8, 3/4, 5/4, 5/4, 5/4, 5/4, (4 + \sqrt{5})/4, (9 + \sqrt{17})/8$
$\Delta_{\mathcal{Q}_{1,1}}$	$0, 1, 1, 1, 1, 11/10, (25 - \sqrt{5})/20, 6/5, 6/5, (25 + \sqrt{5})/20$
$\Delta_{\mathcal{Q}_{2,1}}$	$0, 9/10, 14/15, 14/15, (61 - \sqrt{5})/60, 1, 1, 1, 1, (61 + \sqrt{5})/60$
$\Delta_{\mathcal{Q}_{1,2}}$	$0, (70 - \sqrt{5})/80, (70 + \sqrt{5})/80, (151 - \sqrt{17})/160,$ $15/16, 15/16, 15/16, 15/16, (151 + \sqrt{17})/160, 81/80$

Another way to construct isospectral pairs of regular Kähler graphs is to construct Kähler graphs of product type by using regular graphs. By Propositions 2, 3, 4 we have the following.

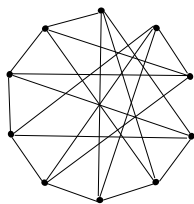


Figure 1

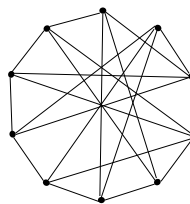


Figure 2

COROLLARY 4. *If we take two isospectral pairs of finite regular graphs  $G_1, G_2$  and  $H_1, H_2$  having the same degrees, then their Kähler graphs of Cartesian product type  $G_1 \hat{\square} H_1, G_2 \hat{\square} H_2$ , of strong product type  $G_1 \hat{\boxtimes} H_1, G_2 \hat{\boxtimes} H_2$ , of semi-tensor product type  $G_1 \hat{\otimes} H_1, G_2 \hat{\otimes} H_2$ , and of lexicographic product type  $G_1 \triangleright H_1, G_2 \triangleright H_2$  are  $(p, q)$ -isospectral pairs for all  $(p, q)$ .*

COROLLARY 5. *Let  $H_1, H_2$  be an isospectral pair of regular graphs having the same degree and  $G$  be a finite graph. Then the Kähler graphs  $G \triangleright H_1, G \triangleright H_2$  of lexicographic product type are  $(p, q)$ -probabilistic transitionary isospectral for all  $(p, q)$ .*

COROLLARY 6. *Let  $H_1, H_2$  be an isospectral pair of regular graphs having the same degree, and let  $G_1, G_2$  be finite graphs satisfying  $n_{G_1} = n_{G_2}$ . Then their Kähler graphs  $G_1 \triangleright H_1, G_2 \triangleright H_2$  of lexicographic product type are  $(p, q)$ -probabilistic transitionary isospectral when  $q$  is odd.*

EXAMPLE 2. We take the regular graphs  $G_1, G_2$  given in Example 1. Let  $H$  be a complete graph with  $n_H = 3$ . Then the pairs  $G_1 \hat{\square} H, G_2 \hat{\square} H$ ;  $G_1 \hat{\boxtimes} H, G_2 \hat{\boxtimes} H$ ;  $G_1 \hat{\otimes} H, G_2 \hat{\otimes} H$ ;  $G_1 \triangleright H, G_2 \triangleright H$  are  $(p, q)$ -isospectral. Since it is known that we do not have isospectral pairs of regular graphs whose cardinalities of their sets of vertices are less than 10, they are the smallest examples of isospectral pairs of regular Kähler graphs obtained by product operations.

In order to study adjacency by  $p$ -step paths without back-tracking Kotani-Sunada [5] consider line graphs (see also [3]). Since Kähler graphs are compound graphs and we consider transition operators for auxiliary graphs, we can not apply their consideration directly to our case. We shall discuss this in future. For a systematic way to construct isospectral pairs of graphs see [7, 8].

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