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Topological Obstruction to Approximating the Irrational Rotation C*-algebra by Certain Fourier Invariant C*-subalgebras

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Abstract. We demonstrate, in a rather quantitative manner, the existence of topological obstructions to approximating the irrational rotation C*-algebra A_θ by Fourier invariant unital C*-subalgebras of either of the forms

$$M \oplus B \oplus \sigma(B), \qquad M \oplus N \oplus D \oplus \sigma(D) \oplus \sigma^2(D) \oplus \sigma^3(D),$$

where M, N are Fourier invariant matrix algebras (over $\mathbb{C}$), B is a C*-subalgebra whose unit projection is flip invariant and orthogonal to its Fourier transform, and D is a C*-subalgebra whose unit projection is orthogonal to its orbit under the Fourier transform. Here, σ is the noncommutative Fourier transform automorphism of A_θ defined by $\sigma(U) = V^{-1}$, $\sigma(V) = U$ on the canonical unitary generators U, V obeying the unitary Heisenberg commutation relation $VU = e^{2\pi i\theta}UV$.

Résumé. On montre l'existence d'obstructions topologiques à l'approximation du tore non-commutatif par sous-algèbres de certains types qui sont invariantes sous l'automorphisme de Fourier.

1. **Introduction** The irrational rotation C*-algebra A_θ (also known as the noncommutative 2-torus) is the unique C*-algebra generated by two unitaries U, V subject to the Heisenberg commutation relation $VU = e^{2\pi i\theta}UV$. (Here, θ is any irrational number in $(0, 1)$.) The noncommutative Fourier transform is the canonical order 4 automorphism σ defined by

$$\sigma(U) = V^{-1}, \quad \sigma(V) = U.$$

In [8] we showed that there is a topologically equivalent order 4 automorphism $\tilde{\sigma}$ of the irrational rotation C*-algebra that models the Fourier transform, in the sense that both agree on $K_1(A_\theta) = \mathbb{Z}^2$, and that $\tilde{\sigma}$ is an inductive limit

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automorphism. More specifically, we proved that the irrational rotation C*-algebra is the closure of an increasing union of unital C*-subalgebras of the form $M \oplus N \oplus B \oplus \tilde{\sigma}(B)$, where M, N are $\tilde{\sigma}$-invariant matrix algebras and B is circle algebra that is $\tilde{\sigma}^2$-invariant. This result suggests that the same may be the case of the Fourier transform σ, but this is still an outstanding problem raised by George Elliott in the early 1990s. The inductive limit problem for the flip (i.e., σ^2) was solved in [5] thanks to the Elliott-Evans structure theorem [2].

The model might suggest that other variations on the approximating basic building blocks may be possible, such as for example invariant unital C*-subalgebras of either of the forms

$$M \oplus B \oplus \sigma(B), \qquad M \oplus N \oplus D \oplus \sigma(D) \oplus \sigma^2(D) \oplus \sigma^3(D)$$

where M, N are Fourier invariant matrix algebras (over $\mathbb{C}$), and B, D appropriate C*-subalgebras orthogonal to their Fourier transforms as indicated in the abstract. We show, however, in this paper that there are certain topological obstructions to approximating the irrational rotation C*-algebra by these forms in a Fourier invariant way. Therefore, our result is the following.

Theorem 1.1. The irrational rotation C*-algebra A_θ cannot be approximated by Fourier invariant unital C*-subalgebras of either of the forms

$$M \oplus B \oplus \sigma(B) \tag{1.1}$$

$$M \oplus N \oplus D \oplus \sigma(D) \oplus \sigma^2(D) \oplus \sigma^3(D) \tag{1.2}$$

where $M \cong M_p(\mathbb{C})$ and $N \cong M_q(\mathbb{C})$ are Fourier invariant matrix algebras, and B, D are C*-subalgebras where the unit projection of B is flip invariant and Fourier orthogonal and the unit projection of D is orthogonal to its Fourier transform and to its flip.

We now present a few basic preliminaries for the proof of this result (given in the next section).

A map $\psi : B \to \mathbb{C}$ on a *-algebra B, with an automorphism σ acting on it, is called a σ-trace when it satisfies the twisted trace condition

$$\psi(xy) = \psi(\sigma(y)x)$$

for $x, y \in B$.

Recall the canonical smooth dense *-subalgebra A_θ^∞ of A_θ consisting of elements x such that its canonical derivations exist to all orders (thus, $\delta_1^m \delta_2^n(x)$ exist in A_θ for all positive integers m, n, where δ_1, δ_2 are the canonical partial derivatives with respect to U and V, respectively.) Equivalently, it consists of infinite series $\sum_{m,n} a_{mn} U^m V^n$ with rapidly decreasing Fourier coefficients.

In [6] we found that there are basically two σ-trace linear maps $\psi_{10}, \psi_{11} : A_\theta^\infty \to \mathbb{C}$ given on the basis elements by

$$\psi_{10}(U^m V^n) = e(-\tfrac{\theta}{4}(m+n)^2)\, \delta_2^{m-n},$$
$$\psi_{11}(U^m V^n) = e(-\tfrac{\theta}{4}(m+n)^2)\, \delta_2^{m-n-1},$$

where the divisor δ-function δ_d^m is 1 when $d|m$ and 0 otherwise, and $e(t) := e^{2\pi i t}$. (Thus, any σ-trace linear map on A_θ^∞ is a linear combination of these two.) In addition, there are three basic σ-invariant σ^2-trace linear maps $\psi_{2j}, j = 0, 1, 2$, given by

$$\psi_{20}(U^m V^n) = e(-\tfrac{\theta}{2}mn)\, \delta_2^m \delta_2^n,$$
$$\psi_{21}(U^m V^n) = e(-\tfrac{\theta}{2}mn)\, \delta_2^{m-1} \delta_2^{n-1},$$
$$\psi_{22}(U^m V^n) = e(-\tfrac{\theta}{2}mn)\, \delta_2^{m-n-1}.$$

When restricted to the fixed point smooth *-subalgebra $A_\theta^{\sigma,\infty} = A_\theta^\infty \cap A_\theta^\sigma$, these functionals define unbounded traces. These unbounded traces ψ_{jk} along with the canonical bounded trace τ comprise the associated Connes-Chern character group homomorphism on the K_0-group of the fixed point C*-subalgebra A_θ^σ:

$$K_0(A_\theta^\sigma) \to \mathbb{C}^6, \quad x \to (\tau(x);\ \psi_{jk}(x)).$$

This map was shown to be injective in [7] for a dense G_δ set of θ's, but since by [1] or [3] one has $K_0(A_\theta^\sigma) \cong \mathbb{Z}^9$ for all θ, this map is injective for all irrational θ. As the maps ψ_{jk} induce group homomorphisms $K_0(A_\theta^\sigma) \to \mathbb{C}$, we simply write $\psi_{1k}(e) := \psi_{1k}[e]$ for any projection e in A_θ.

We shall find it useful to consider the order 2 automorphism γ of A_θ defined by $\gamma(U) = -U, \gamma(V) = -V$ because it commutes with the Fourier transform. It further has the property that it reverses the sign of the ψ_{11}, ψ_{22} invariants while preserving the others: $\psi_{jj}\gamma = -\psi_{jj}$ $(j = 1, 2)$. Thus, if f is a Fourier invariant projection, then so is $\gamma(f)$ and

$$\psi_{10}\gamma(f) = \psi_{10}(f), \quad \psi_{11}\gamma(f) = -\psi_{11}(f),$$

$$\psi_{20}\gamma(f) = \psi_{20}(f), \quad \psi_{21}\gamma(f) = \psi_{21}(f), \quad \psi_{22}\gamma(f) = -\psi_{22}(f). \tag{1.3}$$

In [10] we constructed a canonical Fourier invariant projection of trace $q^2\theta - pq$ for each pair of rational approximants p/q of θ – i.e., such that $0 < \theta - \frac{p}{q} < \frac{1}{q^2}$. Applying this construction with $p = 0, q = 1$ one obtains a Fourier invariant projection e of trace θ (assuming $0 < \theta < 1$). Further, one checks that the topological invariants of this projection are

$$\psi_{10}(e) = \psi_{11}(e) = \tfrac{1-i}{2}, \quad \psi_{20}(e) = \psi_{21}(e) = \tfrac{1}{2}, \quad \psi_{22}(e) = 1. \tag{1.4}$$

2. The Proof

We now prove Theorem 1.1.

PROOF. First, let us note that there exists a Fourier invariant smooth projection e in A_θ such that $\psi_{11}(e) \neq 0$. For example, the projection cited in (1.4) will do.

Assume that A_θ can be approximated by unital C*-subalgebras of the form $M \oplus B \oplus \sigma(B)$ as in (1.1) where the matrix algebra M is σ-invariant and B is a flip invariant C*-subalgebra (thus, $B \oplus \sigma(B)$ is σ-invariant). The matrix units $\{e_{jk}\}$ of M can be chosen to be covariant with respect to σ in the sense that $\sigma(e_{jk}) = \omega_{jk} e_{jk}$ where $\omega_{jk} = \pm 1, \pm i$. (This is because σ is an inner automorphism of M and is therefore given by Ad_w for some order 4 unitary matrix w in M.) Applying the methods of Sakai's Lemma 2 in [4], these matrix units can be approximated by a system of matrix units consisting of smooth or analytic elements. An examination of Sakai's proof shows that the constructed analytic system of matrix units can be chosen to be covariant with respect to σ – thus, the condition $\sigma(e_{jk}) = \omega_{jk} e_{jk}$ can be maintained for smooth matrix units $\{e_{jk}\}$. Therefore, the matrix algebra M can be chosen to be contained in A_θ^∞ and remain Fourier invariant.

Now since $\sigma(x) = wxw^*$ for $x \in M$, where $w \in M$ is an order 4 unitary, it is easy to see that the map

$$\varphi(x) = \mathbf{Tr}(xw^*)$$

defines a σ-invariant σ-trace on M, where $\mathbf{Tr}$ is the unique normalized trace on the matrix algebra M. Further, φ is unique up to scalar multiples. (The uniqueness of φ follows from the uniqueness of the normalized trace on a matrix algebra.)[1] Therefore, now that M is smooth, we can restrict the σ-traces ψ_{10}, ψ_{11} to M and thereby obtain constants c_0, c_1 such that

$$\psi_{10} = c_0 \varphi, \qquad \psi_{11} = c_1 \varphi$$

on M.

(We note that the smooth perturbation of M above can be carried out so that the unit projection g of B remains σ-orthogonal to the unit projection of M and to $\sigma(g)$.)

In view of this, we can approximate the projection e by the Fourier invariant projection

$$e' = e_0 + g_0 + \sigma(g_0),$$

say $\|e - e'\| < 1$, where $e_0 \in M$ is a Fourier invariant projection and g_0 is a flip invariant projection in B such that $g_0 \sigma(g_0) = 0$. The latter equality clearly gives $\psi_{1k}(g_0) = \psi_{1k}(g_0 g_0) = \psi_{1k}(\sigma(g_0) g_0) = 0$. As there is a Fourier invariant unitary

[1] There is a one-to-one correspondence between (σ-invariant) σ-traces on M and trace maps on M given by the relation $\varphi(x) = \mathbf{Tr}(xw^*)$.

v such that $e = ve'v^*$, we have (for $k = 0, 1$)

$$\begin{aligned}\psi_{1k}(e) &= \psi_{1k}(ve'v^*) = \psi_{1k}(e') \\ &= \psi_{1k}(e_0 + g_0 + \sigma(g_0)) \\ &= \psi_{1k}(e_0) = c_k\varphi(e_0).\end{aligned}$$

Writing $1 = h + g + \sigma(g)$ where h is the unit projection of M and g the unit projection of B, we have

$$\psi_{1k}(1) = \psi_{1k}(h) = c_k\varphi(h).$$

Therefore,

$$1 = \psi_{10}(1) = c_0\varphi(h), \qquad 0 = \psi_{11}(1) = c_1\varphi(h).$$

The first of these gives $\varphi(h) \neq 0$, so the second gives $c_1 = 0$. Therefore we have

$$0 \neq \psi_{11}(e) = c_1\varphi(e_0) = 0,$$

a contradiction.

Now assume that A_θ can be approximated by unital C*-subalgebras of the form $M \oplus N \oplus \sigma^*(D)$ as in (1.2), where $\sigma^*(D) := D \oplus \sigma(D) \oplus \sigma^2(D) \oplus \sigma^3(D)$.

By the preceding proof we can similarly arrange to have M, N, D contained in the smooth *-subalgebra A_θ^∞, as would their associated unit projections which we denote by p_1, p_2, p_3, respectively. Thus $1 = p_1 + p_2 + \sigma^*(p_3)$, where we write $\sigma^*(p_3) := p_3 + \sigma(p_3) + \sigma^2(p_3) + \sigma^3(p_3)$.

Restriction of ψ_{2j} to M gives $\psi_{2j} = c_j\varphi_1$ on M where φ_1 is the unique (up to scalars) σ-invariant σ^2-trace on the matrix algebra M. Similarly, restriction of ψ_{2j} to N gives $\psi_{2j} = d_j\varphi_2$ on N where φ_2 is the unique (up to scalars) σ-invariant σ^2-trace on N.

Since $\psi_{2j}(p_3) = 0$, we have

$$\psi_{2j}(1) = \psi_{2j}(p_1) + \psi_{2j}(p_2) = c_j\varphi_1(p_1) + d_j\varphi_2(p_2)$$

($j = 0, 1, 2$) which we write into a matrix equation

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}c_0 & d_0\\ c_1 & d_1\\ c_2 & d_2\end{bmatrix}\begin{bmatrix}\varphi_1(p_1)\\ \varphi_2(p_2)\end{bmatrix}.$$

Let e denote the projection in A_θ^∞ cited at the end of the Introduction whose ψ_{2*}-topological invariants are given by (1.4), so that the projection $\gamma(e)$ has invariants

$$\psi_{20}(\gamma(e)) = \psi_{21}(\gamma(e)) = \tfrac{1}{2}, \quad \psi_{22}(\gamma(e)) = -1$$

in view of (1.3). Approximating $e \approx e'$ and $\gamma(e) \approx f'$ (say to within norm less than 1) by Fourier invariant projections

$$e' = e_1 + e_2 + \sigma^*(e_3), \qquad f' = f_1 + f_2 + \sigma^*(f_3),$$

where e_1, e_2, e_3 are (smooth) projections in M, N, D, respectively, as well as f_1, f_2, f_3, and taking ψ_{2j} of e and of $\gamma(e)$ gives

$$\psi_{2j}(e) = \psi_{2j}(e') = \psi_{2j}(e_1) + \psi_{2j}(e_2) = c_j\varphi_1(e_1) + d_j\varphi_2(e_2)$$
$$\psi_{2j}(\gamma(e)) = \psi_{2j}(f') = \psi_{2j}(f_1) + \psi_{2j}(f_2) = c_j\varphi_1(f_1) + d_j\varphi_2(f_2)$$

(for $j = 0, 1, 2$), or

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} c_0 & d_0 \\ c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} \varphi_1(e_1) & \varphi_1(f_1) \\ \varphi_2(e_2) & \varphi_1(f_1) \end{bmatrix}.$$

Combining the preceding two matrix equations yields the matrix equation over $\mathbb{C}$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} c_0 & d_0 \\ c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} \varphi_1(p_1) & \varphi_1(e_1) & \varphi_1(f_1) \\ \varphi_1(p_2) & \varphi_2(e_2) & \varphi_1(f_1) \end{bmatrix}.$$

This gives a contradiction since the matrix on the left is invertible (its determinant is -1) while the right side is not invertible. This proves Theorem 1.1. $\square$

REMARK 2.1. The proof shows that there are two special projections in the rotation C*-algebra that cannot be simultaneously approximated by unital C*-subalgebras of the form in Theorem 1.1.

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