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The Parallelism of a Certain Tensor of Real Hypersurfaces in a Nonflat Complex Space Form

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Abstract. In Theorem 1, we show a new condition for a real hypersurface M isometrically immersed into a nonflat complex space form to be a hypersurface of type (A). This condition is expressed by the parallelism of a certain tensor of type (1, 1) on M. Furthermore, using the discussion in the proof of Theorem 1, we can give a condition for a Kähler manifold to be a complex space form (see Theorem 2).

Résumé. Dans le théorème 1, nous donnons une nouvelle condition pour qu'une hypersurface réelle M immergée dans une "space form" complexe non plate soit une hypersurface de type (A). Cette condition est exprimée par le parallélisme d'un certain tenseur de type (1, 1) sur M. De plus, en utilisant la discussion dans la démonstration du théorème 1, nous donnons une condition pour qu'une variété de Kähler soit une "space form" complexe (voir le théorème 2).

1. **Introduction** We denote by $\widetilde{M}_n(c)$ a complex $n(\geqq 2)$-dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature c, namely a complex space form. It is well-known that $\widetilde{M}_n(c)$ is holomorphically isometric to either an n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c, a complex Euclidean space $\mathbb{C}^n$ or an n-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c according as c is positive, zero or negative.

The geometry of real hypersurfaces of $\widetilde{M}_n(c)$, $c \neq 0$ is a bit complicated. For examples, the following three properties are known.

(i) There exist no real hypersurfaces M with parallel shape operator A (cf. [7]).
(ii) There exist no real hypersurfaces M with parallel Ricci tensor S (cf. [7]).
(iii) There exist no real hypersurfaces M with parallel structure tensor ϕ (see Proposition 1).

As we mention in Section 2, by the complex structure J of $\widetilde{M}_n(c)$ the structure tensor ϕ is defined on real hypersurfaces M of $\widetilde{M}_n(c)$. We here define a symmetric

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tensor $\psi = \phi A - A\phi$ on M. Using the parallelism of this tensor ψ, we shall characterize hypersurfaces of type (A), which are the simplest examples of Hopf hypersurfaces with constant principal curvatures in a nonflat complex space form.

THEOREM 1. *Let M^{2n-1} $(n \geqq 2)$ be a connected real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ through an isometric immersion. Then M is locally congruent to a hypersurface of type* (A) *if and only if the tensor $\psi = \phi A - A\phi$ is parallel, where ϕ and A are the structure tensor and the shape operator on M, respectively.*

Using the discussion in the proof of Theorem 1, we can prove Theorem 2.

THEOREM 2. *Let $\widetilde{M}$ be a complex $n(\geqq 2)$-dimensional Kähler manifold. Then the following two conditions are mutually equivalent.*

1. *$\widetilde{M}$ is locally congruent to a complex space form.*
2. *At any point $m \in \widetilde{M}$, the tensor $\psi_{m,r} = \phi_{m,r}A_{m,r} - A_{m,r}\phi_{m,r}$ on every sufficiently small geodesic sphere $G_m(r)$ of M is parallel in the direction of the characteristic vector $\xi_{m,r}$ of $G_m(r)$, i.e., $\nabla_{\xi_{m,r}}\psi_{m,r} = 0$, where $\phi_{m,r}$ and $A_{m,r}$ are the structure tensor and the shape operator on $G_m(r)$, respectively in the ambient space $\widetilde{M}$.*

2. Preliminaries Let $\widetilde{M}_n$ be a complex $n(\geqq 2)$-dimensional Kähler manifold with complex structure J and M^{2n-1} a real hypersurface isometrically immersed into $\widetilde{M}_n$. A real hypersurface M^{2n-1} has locally a unit normal vector field $\mathcal{N}$. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n$ and ∇ of M are related by the following formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \tag{2.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{2.2}$$

for arbitrary vector fields X and Y on M, where g is the Riemannian metric induced from the metric of the ambient space $\widetilde{M}_n$ and A is the shape operator of M in $\widetilde{M}_n$. It is well-known that M has an *almost contact metric structure* (ϕ, ξ, η, g) induced from the Kähler structure J of the ambient space $\widetilde{M}_n$. This quadruple is defined by

$$g(\phi X, Y) = g(JX, Y),\ \xi = -J\mathcal{N} \text{ and } \eta(X) = g(\xi, X) = g(JX, \mathcal{N}).$$

The vector field ξ is called *the characteristic vector field* on M. This quadruple satisfies

$$\phi^2 = -I + \eta \otimes \xi,\ g(\xi, \xi) = 1 \quad \text{and} \quad \phi\xi = 0 \tag{2.3}$$

It follows from (2.1), (2.2), $\widetilde{\nabla}J = 0$ and $JX = \phi X + \eta(X)\mathcal{N}$ that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.4}$$

$$\nabla_X \xi = \phi AX. \tag{2.5}$$

We usually call M a *Hopf hypersurface* in $\widetilde{M}_n$ if the characteristic vector field ξ of M is a principal curvature vector at each point of M. For a Hopf hypersurface M we set $A\xi = \delta\xi$ on M.

In the following, we consider a nonflat complex space form $\widetilde{M}_n(c)$ as the ambient space $\widetilde{M}_n$. The following lemma is fundamental (see [7]).

Lemma 1. *Let M^{2n-1} $(n \geqq 2)$ be a Hopf hypersurface with $A\xi = \delta\xi$ in $\widetilde{M}_n(c)$, $c \neq 0$. Then, the following hold:*

(1) *If X is a principal curvature vector of M perpendicular to ξ with $AX = \lambda X$, then $(2\lambda - \delta)A\phi X = (\delta\lambda + (c/2))\phi X$.*

(2) *δ is locally constant on M.*

Remark 1. When $c < 0$, we need to consider the case of $2\lambda - \delta = \delta\lambda + (c/2) = 0$. In fact, the horosphere HS in $\mathbb{C}H^n(c)$ has two distinct constant principal curvatures $\lambda = \sqrt{|c|}/2$ and $\delta = \sqrt{|c|}$.

We here recall the classification theorems of Hopf hypersurfaces M^{2n-1} all of whose principal curvatures are constant on M in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$ (cf. [5,6]).

In $\mathbb{C}P^n(c)$ $(n \geqq 2)$, every Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following:

(A_1) A geodesic sphere $G(r)$ of radius r, where $0 < r < \pi/\sqrt{c}$;
(A_2) A tube of radius r around a totally geodesic $\mathbb{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \pi/\sqrt{c}$;
(B) A tube of radius r around a complex hyperquadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c}\,)$;
(C) A tube of radius r around the Segre embedding of $\mathbb{C}P^1(c)\times\mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and n $(\geqq 5)$ is odd;
(D) A tube of radius r around the Plüker embedding of a complex Grassmannian $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 9$;
(E) A tube of radius r around a Hermitian symmetric space SO(10)/U(5), where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 15$.

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). Unifying real hypersurfaces of types (A_1) and (A_2), we call them hypersurfaces of type (A). The numbers of distinct principal curvatures of these real hypersurfaces are $2, 3, 3, 5, 5, 5$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}P^n(c)$ are given as follows (cf. [8]):

	(A_1)	(A_2)	(B)	(C, D, E)
λ_1	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r)$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r)$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r-\frac{\pi}{4})$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r-\frac{\pi}{4})$
λ_2	—	$-\frac{\sqrt{c}}{2}\tan(\frac{\sqrt{c}}{2}r)$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r+\frac{\pi}{4})$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r+\frac{\pi}{4})$
λ_3	—	—	—	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r)$
λ_4	—	—	—	$-\frac{\sqrt{c}}{2}\tan(\frac{\sqrt{c}}{2}r)$
δ	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$

In $\mathbb{C}H^n(c)$ $(n \geqq 2)$, every Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following:

(A_0) A horosphere HS in $\mathbb{C}H^n(c)$;
$(\mathrm{A}_{1,0})$ A geodesic sphere $G(r)$ of radius r $(0 < r < \infty)$;
$(\mathrm{A}_{1,1})$ A tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;
(A_2) A tube of radius r around a totally geodesic $\mathbb{C}H^{\ell}(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \infty$;
(B) A tube of radius r around a totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$.

These real hypersurfaces are said to be of types (A_0), (A_1), (A_1), (A_2) and (B). Here, type (A_1) means either type $(\mathrm{A}_{1,0})$ or type $(\mathrm{A}_{1,1})$. Unifying real hypersurfaces of types (A_0), (A_1) and (A_2), we call them hypersurfaces of type (A). A real hypersurface of type (B) with radius $r = (1/\sqrt{|c|})\log_e(2+\sqrt{3})$ has two distinct constant principal curvatures $\lambda_1 = \delta = \sqrt{3|c|}/2$ and $\lambda_2 = \sqrt{|c|}/(2\sqrt{3})$. Except for this real hypersurface, the numbers of distinct principal curvatures of Hopf hypersurfaces with constant principal curvatures are $2, 2, 2, 3, 3$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}H^n(c)$ are given as follows:

	(A_0)	$(\mathrm{A}_{1,0})$	$(\mathrm{A}_{1,1})$	(A_2)	(B)
λ_1	$\frac{\sqrt{\lvert c\rvert}}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$
λ_2	—	—	—	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$
δ	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}\,r)$

Next, concerning to the fact (iii) in Introduction, we need the following.

PROPOSITION 1. *There does not exist real hypersurfaces with parallel structure tensor ϕ in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$.*

PROOF. Suppose that there exists a real hypersurface with parallel structure tensor ϕ in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. Then it follows from (2.4) that

$$\eta(Y)AX - g(AX,Y)\xi = 0 \quad \text{for } \forall X, Y \in TM. \tag{2.6}$$

Putting $X = Y = \xi$ in (2.6), we get $A\xi = g(A\xi, \xi)\xi$, so that our real hypersurface M is a Hopf hypersurface. Next, we take a principal curvature vector X orthogonal to ξ with $AX = \lambda X$. Then, setting $Y = \xi$ in (2.6), we can see that the principal curvature vector X satisfies $AX = 0$, i.e., $\lambda = 0$. Hence our real hypersurface M is a Hopf hypersurface having two distinct constant principal curvature δ and $\lambda = 0$. However there does not exist such a Hopf hypersurface (see the above tables of principal curvatures), which is a contradiction. Therefore we obtain the desired statement. □

For the proof of Theorem 2, we prepare the following lemma due to Chen and Vanheche (see [3]).

LEMMA 2. *Let M be a Riemannian manifold of dimension greater than two with Riemannian metric g. We denote by $G_m(r)$ a geodesic sphere with center m and radius r in M, and by $A_{m,r}$ the shape operator of $G_m(r)$ in M with respect to the outward unit normal vector field. For non-zero tangent vectors $v, w \in T_mM$ at a point $m \in M$, we choose a unit tangent vector $u \in T_mM$ orthogonal to both v and w. We denote by $v_r, w_r \in T_{\exp_m(ru)}M$ the parallel displacement of v, w along the geodesic segment $exp_m(su)$, $0 \leqq s \leqq r$. Then for sufficiently small r we have*

$$g(A_{m,r}v_r, w_r) = \frac{1}{r}\, g(v,w) - \frac{r}{3}\, g(R(u,v)w,u) + O(r^2), \tag{2.7}$$

where R is the Riemannian curvature tensor defined by $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$.

3. Proof of Theorem 1

We first mention the following well-known proposition without proof (see [6, 7]).

PROPOSITION 2. *Let M^{2n-1} ($n \geqq 2$) be a connected real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ through an isometric immersion. Then the following two conditions are mutually equivalent.*

(1) *M is locally congruent to a hypersurface of type* (A).
(2) *$\phi A = A\phi$, i.e., $\psi = \phi A - A\phi = 0$ on M.*

We first suppose that our real hypersurface M is locally congruent to a hypersurface of type (A). Then from Proposition 2, we know $\psi = 0$, so that ψ is parallel in a trivial sense.

Next, we suppose that ψ is parallel on M. The following discussion is essentially due to the work of [4]. By direct computation, from (2.4) we get the following:

$$(3.1)\qquad \begin{aligned} 0 = g((\nabla_X\psi)Y, Z) &= \eta(AY)g(AX, Z) - \eta(Z)g(AX, AY) \\ &\quad + g((\phi(\nabla_X A) - (\nabla_X A)\phi)Y, Z) \\ &\quad + \eta(AZ)g(AX, Y) - \eta(Y)g(AX, AZ) \end{aligned}$$

for any tangent vector fields X, Y and Z on M. Putting $X = Y = Z = \xi$ in (3.1), because of (2.3) we obtain

$$g((\nabla_\xi\psi)\xi, \xi) = 2(\eta(A\xi)^2 - \eta(A^2\xi)).$$

So we have

$$(3.2)\qquad \eta(A\xi)^2 - \eta(A^2\xi) = 0$$

Setting $A\xi = a\xi + bU$ for two functions a and b on M, where U is the unit tangent vector field orthogonal to ξ on M and substituting this in (3.2), we get $a^2 - (a^2 + b^2) = 0$. So, we conclude that $b = 0$, i.e., ξ is a principal curvature vector. Namely, our real hypersurface is a Hopf hypersurface.

Next, we take a principal curvature vector X with principal curvature λ orthogonal to ξ. Setting $Y = X, Z = \xi$ in (3.1) and multiplying $2\lambda - \delta$ to (3.1), from Lemma 1, (2.3), (2.4), (2.5), we find

$$\begin{aligned} 0 &= (2\lambda - \delta)g((\nabla_X\psi)X, \xi) \\ &= (-\lambda^2 + \delta\lambda)(2\lambda - \delta)g(X, X) - (2\lambda - \delta)g(\phi X, (\nabla_X A)\xi) \\ &= (-\lambda^2 + \delta\lambda)(2\lambda - \delta)g(X, X) + (2\lambda - \delta)(g(\delta\phi^2 X, AX) + g(A\phi X, \phi AX)) \\ &= (-\lambda^2 + \delta\lambda)(2\lambda - \delta)g(X, X) \\ &\qquad + (2\lambda - \delta)(g(\delta(-X + \eta(X)\xi), AX) + g((\delta\lambda + (c/2))\phi X, \phi AX) \\ &= -\lambda^2(2\lambda - \delta)g(X, X) - (\delta\lambda + (c/2))g(\phi^2 X, AX) \\ &= -\lambda(2\lambda^2 - 2\delta\lambda - (c/2))g(X, X), \end{aligned}$$

so that $\lambda(2\lambda^2 - 2\delta\lambda - (c/2)) = 0$. We here note that $\lambda = 0$ is *not* a solution to the quadratic equation $2\lambda^2 - 2\delta\lambda - (c/2) = 0$. Hence our Hopf hypersurface M in $\widetilde{M}_n(c)$ has at most four constant principal curvatures δ, $\lambda = 0$ and λ_1, λ_2 which are solutions to the equation $2\lambda^2 - 2\delta\lambda - (c/2) = 0$. However we emphasize $\lambda \neq 0$ (see the above tables of principal curvatures). So we see that our Hopf hypersurface M has at most three constant principal curvatures δ and λ_1, λ_2 which are solutions to the following quadratic equation:

$$2\lambda^2 - 2\delta\lambda - (c/2) = 0. \tag{3.3}$$

When our Hopf hypersurface M with constant principal curvatures does not have a principal curvature λ with $2\lambda - \delta = 0$, Equation (3.3) can be rewritten as:

$$\lambda = \frac{\delta\lambda + (c/2)}{2\lambda - \delta}.$$

This means $\phi AX = A\phi X$ for every X in V_λ, which, together with the fact that $\phi A\xi = 0 = A\phi\xi$, implies $\phi A = A\phi$ on M. Then M is locally congruent to a hypersurface of type (A) in a nonflat complex space form (see Proposition 2).

Finally, we investigate the case that our Hopf hypersurface M with constant principal curvatures has a principal curvature λ with $2\lambda - \delta = 0$. Then M is nothing but the horosphere HS in $\mathbb{C}H^n(c)$ (see the above table of principal curvatures in the case of $c < 0$), so that M is a member of hypersurfaces of type (A). Therefore we obtain the desired conclusion.

4. **Proof of Theorem 2** We first show that Condition (2) implies Condition (1). We denote by $\xi_{m,r}$ and $\eta_{m,r}$ the characteristic vector and the contact form on our geodesic sphere $G_m(r)$ in a Kähler manifold $\widetilde{M}$. By the assumption $\nabla_{\xi_{m,r}}\psi_{m,r} = 0$ we have

$$\eta_{m,r}(A_{m,r}\xi_{m,r})^2 - \eta_{m,r}(A^2_{m,r}\xi_{m,r}) = 0$$

which corresponds to Equation (3.2), so that the geodesic sphere $G_m(r)$ is a Hopf hypersurface in the Kähler manifold $\widetilde{M}$.

Next, in Lemma 2 we choose w orthogonal to both v and Jv and we put $u = Jv$. Since u_r is a normal vector on $G_m(r)$ in $\widetilde{M}$ at $\exp_m(ru)$, the vector v_r is the characteristic vector of $G_m(r)$ at this point. It follows from the fact that our geodesic sphere $G_m(r)$ is a Hopf hypersurface in $\widetilde{M}$ and Equation (2.7) that the curvature tensor R of $\widetilde{M}$ satisfies

$$g(R(u, Ju)w, u) = 0$$

(cf. [1]). Hence we can see that $R(u, Ju)u$ is proportional to Ju for every $u \in T_m\widetilde{M}$, so that $\widetilde{M}$ is locally congruent to a complex space form (see [9]). Thus we obtain Condition (1).

Conversely, we suppose Condition (1). We take an arbitrary geodesic sphere $G(r)$ in a complex space form $\widetilde{M}_n(c)$. Since $G_m(r)$ is a totally umbilic hypersurface in the case of $c = 0$, the tensor $\psi_{m,r} = \phi_{m,r}A_{m,r} - A_{m,r}\phi_{m,r}$ on $G_m(r)$ vanishes. On the other hand, when $c \neq 0$, our geodesic sphere $G_m(r)$ is not totally umbilic. However the tensor $\psi_{m,r}$ on $G_m(r)$ also vanishes (see Proposition 2). Therefore the tensor $\psi_{m,r}$ is parallel in a trivial sense. Thus we obtain Condition (2).

REFERENCES

1. T. Adachi and S. Maeda, *Space forms from the viewpoint of their geodesic spheres*, Bull. Austral. Math. Soc. **62** (2000), 205–210.
2. J. Berndt, *Real hypersurfaces with constant principal curvatures in complex hyperbolic space*, J. Reine Angew. Math. **395** (1989), 132–141.
3. B. -y. Chen and L. Vanheche *Differential geometry of geodesic spheres*, J. Reine Angew. Math. **325** (1981), 28–67.
4. T. Hamada, *Some real hypersurfaces of complex projective space*, Proc. of the Workshop on Differential Geometry of Submanifolds and its Related Topics. Saga, August 4-6, 2012, 82–86, World Scientific.
5. M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc. **296** (1986), 137–149.
6. M. Okumura, *On some real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc. **212** (1975), 355–364.
7. R. Niebergall and P.J. Ryan, *Real hypersurfaces in complex space forms*, Tight and Taut Submanifolds, T.E. Cecil and S.S. Chern (eds.), Cambridge Univ. Press, 1998, 233–305.
8. R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10** (1973), 495–506.
9. S. Tanno, *Constancy of holomorphic sectional curvature in almost Hermitian manifolds*, Kodai Math. Sem. Rep. **25** (1973), 190–201.

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