Counting Toric Actions on Symplectic Four-Manifolds

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Abstract. Given a symplectic manifold, we ask in how many different ways can a torus act on it. Classification theorems in equivariant symplectic geometry can sometimes tell that two Hamiltonian torus actions are inequivalent, but often they do not tell whether the underlying symplectic manifolds are (non-equivariantly) symplectomorphic. For two dimensional torus actions on closed symplectic four-manifolds, we reduce the counting question to combinatorics, by expressing the manifold as a symplectic blowup in a way that is compatible with all the torus actions simultaneously. For this we use the theory of pseudoholomorphic curves.

Résumé. Nous nous intéressons aux différentes actions d'un tore sur une variété symplectique donnée. En géométrie symplectique équivariante, les théorèmes de classification permettent parfois de distinguer des actions hamiltoniennes de tores géométriquement inéquivalentes. Par contre, ces théorèmes ne permettent habituellement pas de déterminer si les variétés symplectiques sous-jacentes sont symplectomorphes. Dans le cas des variétés symplectiques de dimension 4, nous réduisons le problème d'énumération des actions toriques inéquivalentes à un problème combinatoire en exprimant la variété considérée comme un éclatement symplectique qui est compatible simultanément avec toutes les actions toriques. Ce résultat est obtenu en employant des techniques pseudo-holomorphes.

1. **Introduction** An **action** of a torus $T \cong (S^1)^k$ on a symplectic manifold (M, ω) is a group homomorphism $\rho: T \to \mathrm{Symp}(M)$ that is smooth in the diffeological sense: the map $(a, m) \mapsto \rho(a)(m) =: a \cdot m$ from $T \times M$ to M is smooth. A **momentum map** for such an action is a map $\Phi: M \to \mathfrak{t}^* \cong \mathbb{R}^k$ such that $d\Phi_j = -\iota(\xi_j)\omega$ for all $j = 1, \ldots, k$, where $\xi_1, \ldots, \xi_k$ are the vector fields that generate the torus action. When considering torus actions on symplectic manifolds, we will always assume that *the action is faithful*, i.e., ρ is one-to-one; that *the manifold M is connected*; and that *the action is Hamiltonian*, i.e., that a momentum map exists. If $\dim T = \frac{1}{2}\dim M$, the Hamiltonian T-action is called **toric**. If, additionally, M is compact, then the image of the momentum map is a unimodular ("Delzant") polytope, and this polytope determines the triple

(M, ω, Φ) up to an equivariant symplectomorphism that respects the momentum map; this is Delzant's theorem [2]. It follows that two toric actions differ by conjugation in $\mathrm{Symp}(M)$ and reparametrization of T if and only if their momentum images are $\mathrm{AGL}(n, \mathbb{Z})$-congruent where $n = \dim T = \frac{1}{2} \dim M$, that is, they differ by a transformation of the form $x \mapsto Ax + b$ where A is in $\mathrm{GL}(n, \mathbb{Z})$ and $b \in \mathbb{R}^n$; we consider such actions as **equivalent**.

Up to equivalence, a compact symplectic four-manifold has only finitely many toric actions. We prove this fact in [9]; the proof uses "soft" equivariant, algebraic, and combinatorial methods. This finiteness result remains true in higher dimensions; this is proved by Borisov and McDuff in [14, Proposition 3.1]. In dimension four, the number of inequivalent Hamiltonian circle actions that do not extend to 2-torus actions is also finite; this is proved by Pinsonnault [16].

If the second Betti number of a compact symplectic four-manifold is one or two, a "soft" argument gives the exact number of inequivalent toric actions; see [6]. But in more general cases the proofs in [9,14,16] that the number of actions is finite do not give us the actual number of nonequivalent toric actions; in general they do not even determine whether this number is nonzero, that is, whether toric actions exist.

Whether toric or circle actions exist was determined for symplectic manifolds that are obtained from a $\mathbb{CP}^2$ of size λ by k symplectic blowups of equal size ϵ, for many values of λ and ϵ, in [7], [11], and [16]: if $k \geq 4$ then the manifold does not admit a 2-torus action, and if $(k-1)\epsilon \geq \lambda$ then the manifold does not admit a circle action. (The "size" of a blowup is $\frac{1}{2\pi}$ times the symplectic area of the exceptional divisor; the "size" of $\mathbb{CP}^2$ is $\frac{1}{2\pi}$ times the symplectic area of $\mathbb{CP}^1 \subset \mathbb{CP}^2$.) The challenge is to show that there are no "exotic actions": every action can be made "consistent with the blow-ups" so that it comes from an action on $\mathbb{CP}^2$. The proofs combine "soft" and "hard" (holomorphic) techniques.

In our more recent work we show that there are no "exotic actions" also for arbitrary blowups of $\mathbb{CP}^2$. In fact, every compact connected symplectic four-manifold that admits a toric action and whose second Betti number is ≥ 3 is (non-equivariantly) symplectomorphic to a symplectic manifold that is obtained from $\mathbb{CP}^2$ with a multiple of the Fubini-Study form by a sequence of symplectic blowups. (This follows from Delzant's theorem [2] and observations of [3, Section 2.5] and [6, Lemma 3]; see [9, Corollary 2.17].) Thus, our work yields an algorithm to count the exact number of inequivalent toric actions on an *arbitrary* compact symplectic four-manifold. In this paper we report on this work, deferring details to a longer and more leisurely exposition [10].

We denote by

$$(M_k, \omega_{\lambda; \delta_1, \ldots, \delta_k})$$

a symplectic manifold that is obtained from a $\mathbb{CP}^2$ of size λ by blowups of sizes $\delta_1, \ldots, \delta_k$. If such a manifold exists, then it is unique up to symplectomorphism, by McDuff's work [13].

To compare different blowups, it is convenient to fix the underlying manifold M_k. Once and for all, we fix a sequence $p_1, p_2, p_3, \ldots$ of distinct points on

the complex projective plane $\mathbb{CP}^2$, and we denote by M_k the manifold that is obtained from $\mathbb{CP}^2$ by complex blowups at $p_1, \ldots, p_k$. We have a decomposition

$$H_2(M_k) = \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_k$$

where L is the image of the homology class of a line $\mathbb{CP}^1$ in $\mathbb{CP}^2$ under the inclusion map $H_2(\mathbb{CP}^2) \to H_2(M_k)$ and where $E_1, \ldots, E_k$ are the homology classes of the exceptional divisors.

Throughout this paper, homology is taken with integer coefficients and cohomology is taken with real coefficients.

A **blowup form** on M_k is a symplectic form for which there exist pairwise disjoint embedded symplectic spheres in the classes $L, E_1, \ldots, E_k$.

Fix a non-negative integer k. Let $\langle \cdot, \cdot \rangle$ denote the pairing between cohomology and homology on M_k. A vector $(\lambda; \delta_1, \ldots, \delta_k)$ in $\mathbb{R}^{1+k}$ **encodes** a cohomology class $\Omega \in H^2(M_k; \mathbb{R})$ if $\frac{1}{2\pi} \langle \Omega, L \rangle = \lambda$ and $\frac{1}{2\pi} \langle \Omega, E_j \rangle = \delta_j$ for $j = 1, \ldots, k$.

Thus, $\omega_{\lambda; \delta_1, \ldots, \delta_k}$ can be taken to be a blowup form on M_k whose cohomology class is encoded by the vector $(\lambda; \delta_1, \ldots, \delta_k)$. It is unique up to a diffeomorphism that acts trivially on the homology, as follows from results of Gromov [4, 2.4.A', 2.4.A1'] and McDuff [12, 13].

Let $k \geq 3$, and let $\lambda, \delta_1, \ldots, \delta_k$ be real numbers. The vector $(\lambda; \delta_1, \ldots, \delta_k)$ is **reduced** if

$$\delta_1 \geq \cdots \geq \delta_k \quad \text{and} \quad \delta_1 + \delta_2 + \delta_3 \leq \lambda.$$

In [8, Theorem 1.4] we show the following result:

THEOREM 1.1. *Let $k \geq 3$. Given a blowup form $\omega_{\lambda'; \delta_1', \ldots, \delta_k'}$ on M_k, there exists a unique reduced vector $(\lambda; \delta_1, \ldots, \delta_k)$ such that $(M_k, \omega_{\lambda'; \delta_1', \ldots, \delta_k'}) \cong (M_k, \omega_{\lambda; \delta_1, \ldots, \delta_k})$.*

Here is our main result about the non-existence of "exotic actions":

THEOREM 1.2. *Let k be an integer ≥ 3. Let $\omega_{\lambda; \delta_1, \ldots, \delta_k}$ be a blowup form on M_k whose cohomology class is encoded by a reduced vector $(\lambda; \delta_1, \ldots, \delta_k)$. Let*

(1.3)
$$\delta = \lambda - \delta_1 - \delta_2,$$
$$a = \lambda - \delta_2,$$
$$b = \lambda - \delta_1.$$

Then every toric action on $(M_k, \omega_{\lambda; \delta_1, \ldots, \delta_k})$ is isomorphic to one that is obtained from a toric action on $(S^2 \times S^2, a\omega_{S^2} \oplus b\omega_{S^2})$ by a sequence of equivariant symplectic blowups of sizes $\delta, \delta_3, \ldots, \delta_k$.

Figure 1: A "Hirzebruch trapezoid" and the effect of an equivariant blowup

Moreover, Theorem 1.2 holds if we replace "toric action" with "action of a (not necessarily connected) compact Lie group G that preserves ω and induces the identity morphism on $H_2(M_k)$".

The (equivalence classes of) toric actions on $(S^2 \times S^2, a\omega_{S^2} \oplus b\omega_{S^2})$ are enumerated by the set of integers ℓ such that $0 \le \ell < \frac{a}{b}$; their momentum map images are the Hirzebruch trapezoids $H_{a,b,2\ell}$, with height b, average width a, left edge vertical, and right edge of slope $-\frac{1}{2\ell}$ or vertical if $\ell = 0$ [6, Theorem 2]; see Figure 1.

The effect of an equivariant blowup of size ϵ on the momentum mapping image Δ amounts to "chopping off a corner of size ϵ", see Figure 1. This can be done at a vertex of Δ if and only if all the edges that emanate from that vertex have size $> \epsilon$. Here, the "size" of a vector of rational slope is the positive number such that the vector is equal to that number times a primitive lattice element.

Given a unimodular ("Delzant") polygon, choose a cyclic ordering of its edges, let $u_1, \ldots, u_N$ be the primitive inner normals to the edges, and let $a_1, \ldots, a_N$ be the sizes of the edges. There exist integers k_j such that $u_{j+1} + u_{j-1} = -k_j u_j$. (If the polygon is the momentum image for a toric action on a symplectic manifold M, the preimages of the edges are embedded symplectic two-spheres in M, the k_j are their self intersections, and the a_j are $\frac{1}{2\pi}$ times their symplectic areas.) Two polygons are $\mathrm{AGL}(2,\mathbb{Z})$-congruent if and only if their corresponding vectors $\{(k_j, a_j)\}$ differ by a cyclic or anti-cyclic permutation.

Therefore, by Theorem 1.2 and Delzant's theorem, we obtain the list of (equivalence classes of) toric actions on $(M_k, \omega_{\lambda; \delta_1, \ldots, \delta_k})$ from the following combinatorial algorithm. Recursively produce all the polygons that are obtained by a sequence of corner choppings of sizes $\delta, \delta_3, \ldots, \delta_k$ starting from a Hirzebruch trapezoid $H_{a,b,2\ell}$, where a, b, δ are as in (1.3) and $0 \le \ell < \frac{a}{b}$. Use the associate vectors to eliminate repetitions up to $\mathrm{AGL}(2,\mathbb{Z})$.

Here is a sample of some consequences of this algorithm. Let k be an integer ≥ 3. Let ω be a blowup form on M_k whose cohomology class is encoded by a reduced vector $(\lambda; \delta_1, \ldots, \delta_k)$.

1. Suppose that
$$\lambda - \delta_1 - \delta_2 = \delta_3 = \delta_4 = \delta_5 = \delta_6,$$
   or that there exists $i \ge 1$ such that
$$\delta_i = \delta_{i+1} = \cdots = \delta_{i+(i+2)}.$$

Then there is no toric action on (M_k, ω).

2. The number of toric actions on (M_k, ω) is at most

$$(1.4) \qquad \left(\left\lceil \frac{\lambda - \delta_2}{\lambda - \delta_1} \right\rceil + \left\lceil \frac{\delta_1 - \delta_2}{\lambda - \delta_1} \right\rceil \right) \cdot \frac{(k+2)!}{24}.$$

This upper bound is achieved if and only if the following conditions (i)–(iv) hold.

(i) $\left\lceil \frac{\lambda - \delta_2}{\lambda - \delta_1} \right\rceil \cdot (\lambda - \delta_1) < \lambda$,

(ii) $\delta_{j+1} + \cdots + \delta_k < \delta_j \quad$ for all $3 \le j < k$,

(iii) $\delta_3 + \cdots + \delta_k < \lambda - \delta_1 - \delta_2, \quad$ and

(iv) $\delta_3 + \cdots + \delta_k < \lambda - \left\lceil \frac{\lambda - \delta_2}{\lambda - \delta_1} \right\rceil \cdot (\lambda - \delta_1)$.

For example, this upper bound is attained for $\lambda = 1, \quad \delta_1 = \delta_2 = \frac{1}{x}$, and $\delta_i = \frac{1}{x^{i-1}} \quad$ for $3 \le i \le k$, where $x \ge \frac{3 + \sqrt{5}}{2}$.

2. Sketch of proof of Theorem 1.2

We use holomorphic tools in almost complex four-manifolds. An **almost complex structure** on a manifold M is an automorphism $J \colon TM \to TM$ such that $J^2 = -\operatorname{id}$. An almost complex structure J on M is **tamed** by a symplectic form ω if $\omega(u, Ju)$ is positive for all nonzero $u \in TM$; let $\mathcal{J}_\tau(M, \omega)$ denote the set of almost complex structures that are tamed by ω. A **J-holomorphic sphere** is a J-holomorphic map from $\mathbb{CP}^1$ to M; it is **simple** if it cannot be factored through a branched covering of the domain. The (genus zero) **Gromov-Witten invariant** of a compact four dimensional manifold (M, ω) associates to every homology class $A \in H_2(M)$ an integer GW(A). Let $\kappa(A) = (A \cdot A + c_1(TM)(A))/2$. For generic $(J, p_1, \ldots, p_{\kappa(A)}) \in \mathcal{J}_\tau(M, \omega) \times M^{\kappa(A)}$, the integer $GW(A)$ is equal to the number of simple J-holomorphic spheres $\mathbb{CP}^1 \to M$ in the class A that pass through the points $p_1, \ldots, p_{\kappa(A)}$, modulo reparametrization by an element of $\mathrm{PSL}(2, \mathbb{C})$, and counted with appropriate signs.

Let $k \ge 3$. Let ω be a blowup form on M_k whose cohomology class is encoded by a vector $(\lambda; \delta_1, \ldots, \delta_k)$ that is reduced.

LEMMA 2.1. *Let A be a class in $H_2(M_k)$. Suppose that $c_1(TM_k)(A) \ge 1$, and suppose that A is represented by a nontrivial J-holomorphic sphere for some almost complex structure J that is tamed by some blowup form on M_k. Then*

$$(2.2) \qquad \frac{1}{2\pi} \langle [\omega], A \rangle \ge \delta_k.$$

PROOF. See [8, Lemma 3.6] or [10]. □

THEOREM 2.3. *Let A be a class in $H_2(M_k)$ such that $c_1(TM_k)(A) > 0$ and GW(A) $\ne 0$. Suppose that*

$$\frac{1}{2\pi} \langle [\omega], A \rangle = \delta_k.$$

Then for every almost complex structure J that is ω-tamed there exists a J-holomorphic sphere in the class A.

Proof.   Let J be an almost complex structure that is ω-tamed. Because $GW(A) \neq 0$, and by Gromov's compactness theorem [4, 1.5.B], there exist classes $A_1, \ldots, A_\ell \in H_2(M_k)$ such that

$$A = A_1 + \cdots + A_\ell$$

and such that each A_j can be represented by a nonconstant J-holomorphic sphere. We claim that $\ell = 1$. Because $c_1(TM_k)(A) > 0$, there exists at least one summand A_j such that $c_1(TM_k)(A_j) > 0$. Fix such an A_j. Then $c_1(TM_k)(A_j) \geq 1$. By Lemma 2.1, the jth summand in the sum $\sum_{i=1}^{\ell} \frac{1}{2\pi} \langle [\omega], A_i \rangle$ is $\geq \delta_k$. But the entire sum is equal to δ_k and all the summands are strictly positive. So $\ell = 1$, as required.   □

A homology class $E \in H_2(M_k)$ is **exceptional** if it is represented by an embedded ω-symplectic sphere with self intersection -1. Note that $E_1, \ldots, E_k$ are exceptional.

Corollary 2.4.   *For every almost complex structure J that is ω-tamed there exists an embedded J-holomorphic sphere in the class E_k.*

Moreover, let E be any exceptional class in $H_2(M_k)$ such that $\frac{1}{2\pi} \langle [\omega], E \rangle = \delta_k$. Then for every almost complex structure J that is ω-tamed there exists an embedded J-holomorphic sphere in the class E.

Proof.   For an exceptional class $E \in H_2(M_k)$, we have $c_1(TM_k)(E) = 1$ and, by McDuff's "C_1 lemma" [12, Lemma 3.1] and Gromov's compactness theorem [4, 1.5.B], the invariant $GW(E)$ is nonzero. Therefore we can apply Theorem 2.3 to E. By the adjunction formula [15, Corollary 1.7], a J-holomorphic sphere in E is embedded.   □

Lemma 2.5.   *Let $E \in H_2(M_k)$ be such that $E \cdot E = -1$. Let a compact Lie group G act on M_k. Suppose that the G action preserves ω and induces the identity morphism on $H_2(M_k)$. Let J_G be a G-invariant ω-tamed almost complex structure on M_k. Let C be an embedded J_G-holomorphic sphere in the class E. Then C is a G-invariant ω-symplectic embedded sphere.*

Proof.   Because J_G is ω-tamed and C is an embedded J_G-holomorphic sphere, C is an embedded ω-symplectic sphere.

Let $a \in G$. Because G acts trivially on the homology, $[aC] = [C] = E$. By positivity of intersections of J-holomorphic spheres in an almost complex four-manifold [15, Proposition 2.4.4], and since $E \cdot E = -1$, the spheres aC and C must coincide. Thus, C is G-invariant.   □

Corollary 2.6.   *Let a compact Lie group G act on M_k, preserve ω, and act trivially on $H_2(M_k)$. Then there exists a G-invariant ω-symplectic embedded sphere C in the class E_k.*

Equivariantly blowing down along the sphere C yields a G-action on $(M_{k-1}, \omega_{\lambda;\delta_1,...,\delta_{k-1}})$. By repeated $k-2$ applications of Corollary 2.6, we reduce Theorem 1.2 to the following claim.

CLAIM 2.7. *Let ω be a blowup form on M_2, whose cohomology class is encoded by a vector $(\lambda; \delta_1, \delta_2)$ with $\delta_1 \geq \delta_2$. Let a compact Lie group G act on M_2, preserve ω, and act trivially on $H_2(M_2)$.*

Then there exists an embedded G-invariant ω-symplectic sphere in the class $L - E_1 - E_2$, and blowing down along this sphere yields a G-action on $(S^2 \times S^2, a\omega_{S^2} \oplus b\omega_{S^2})$ where $a = \lambda - \delta_2$ and $b = \lambda - \delta_1$.

The claim is proved by combinatorial tools in case the action is toric and by holomorphic tools in the general case. See [17, Lemma 2.2] and [10].

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